

# AVERAGE DISTANCES BETWEEN POINTS IN GRAPH-DIRECTED SELF-SIMILAR FRACTALS

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**ABSTRACT.** We study several distinct notions of average distances between points belonging to graph-directed self-similar subsets of  $\mathbb{R}$ . In particular, we compute the average distance with respect to graph-directed self-similar measures, and with respect to the normalised Hausdorff measure.

As an application of our main results, we compute the average distance between two points belonging to the Drobot-Turner set  $T_N(c, m)$  with respect to the normalised Hausdorff measure, i.e. we compute

$$\frac{1}{\mathcal{H}^s(T_N(c, m))^2} \int_{T_N(c, m)^2} |x - y| d(\mathcal{H}^s \times \mathcal{H}^s)(x, y),$$

where  $s$  denotes the Hausdorff dimension of  $T_N(c, m)$  and  $\mathcal{H}^s$  is the  $s$ -dimensional Hausdorff measure; here the Drobot-Turner set (introduced by Drobot & Turner in 1989) is defined as follows, namely, for positive integers  $N$  and  $m$  and a positive real number  $c$ , the Drobot-Turner set  $T_N(c, m)$  is the set of those real numbers  $x \in [0, 1]$  for which any  $m$  consecutive base  $N$  digits in the  $N$ -ary expansion of  $x$  sum up to at least  $c$ . For example, if  $N = 2$ ,  $m = 3$  and  $c = 2$ , then our results show that

$$\frac{1}{\mathcal{H}^s(T_2(2, 3))^2} \int_{T_2(2, 3)^2} |x - y| d(\mathcal{H}^s \times \mathcal{H}^s)(x, y) = \frac{4444\lambda^2 + 2071\lambda + 3030}{12141\lambda^2 + 5650\lambda + 8281} = 0.36610656 \dots,$$

where  $\lambda = 1.465571232 \dots$  is the unique positive real number such that  $\lambda^3 - \lambda^2 - 1 = 0$ .

## 1. INTRODUCTION.

The average distance between two points belonging to a self-similar subset of  $\mathbb{R}$  has recently been investigated in [AlEdHaOl, LeRuHa]. However, it has for some been recognised that while self-similar constructions provide an important framework for studying fractal and multifractal geometry, the more general notion of graph-directed self-similar constructions provide a substantially more flexible and useful framework, see, for example, [MaWi] and the text [MaUr] for a detailed discussion of this. In recognition of this viewpoint, the purpose of this paper is to study the average distance between two points of graph-directed self-similar subsets of  $\mathbb{R}$ .

Let  $(K_i)_{i \in V}$  be the family of graph-directed self-similar sets associated with a graph-directed self-similar iterated function system in  $\mathbb{R}$  associated with a finite directed multigraph  $G = (V, E)$  where  $V$  denotes the set of vertices of  $G$  and  $E$  denotes the set of edges of  $G$ ; the precise definitions will be given in Section 1.1 below. In this paper we compute the “natural geometric” average distance between two

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points in the set  $K_i$ . If  $K_{n,i}$  denotes the  $n$ 'th order approximation to  $K_i$  (the precise definition of  $K_{n,i}$  will be given in Section 1.1), then the number

$$\frac{\int_{K_{n,i}^2} |x - y| d(x, y)}{\int_{K_{n,i}^2} d(x, y)},$$

may be interpreted as the average distance between two points chosen uniformly from  $K_{n,i}$ . We show that the following limiting average distance, namely,

$$\lim_n \frac{\int_{K_{n,i}^2} |x - y| d(x, y)}{\int_{K_{n,i}^2} d(x, y)}, \quad (1.1)$$

exists and we provide an explicit value for it; this is the content of Corollary 2.4.

There is another, and perhaps equally natural, way to define the average distance between two points from  $K_i$ . Namely, the average distance between two points in  $K_i$  chosen with respect to the “natural” uniform distribution on  $K_i$ , i.e. chosen with respect to the normalised Hausdorff measure on  $K_i$ . More precisely, if  $s$  denotes the Hausdorff dimension of  $K_i$  and  $\mathcal{H}^s$  denotes the  $s$ -dimensional Hausdorff measure, then we compute the average distance between two points in  $K_i$  chosen with respect to the normalised  $s$ -dimensional Hausdorff measure on  $K_i$ , i.e. we compute the integral

$$\frac{1}{\mathcal{H}^s(K_i)^2} \int_{K_i^2} |x - y| d(\mathcal{H}^s \times \mathcal{H}^s)(x, y); \quad (1.2)$$

this is the content of Corollary 2.5. Somewhat surprisingly, even for self-similar constructions, the averages in (1.1) and (1.2) do not, in general, coincide; examples of this are given in [AlEdHaOl].

In fact, we compute far more general averages than those in (1.1) and (1.2). Namely, if  $(\mu_{\mathbf{p},i})_i$  and  $(\mu_{\mathbf{q},i})_i$  are the graph-directed self-similar measures on  $K_i$  associated with the vectors  $\mathbf{p}$  and  $\mathbf{q}$  (the precise definitions will be given in Section 1.1), then we compute the average distance between two points in  $K_i$  where the first point is chosen with respect to the measure  $\mu_{\mathbf{p},i}$  and where the second point is chosen with respect to the measure  $\mu_{\mathbf{q},i}$ , i.e. we compute the average distance defined by

$$\int_{K_i^2} |x - y| d(\mu_{\mathbf{p},i} \times \mu_{\mathbf{q},i})(x, y); \quad (1.3)$$

see Theorem 2.1. The averages in (1.1) and (1.2) are, in fact, special cases of the more general average in (1.3). We will now explain this in more detail. Indeed, assume that the sets  $(K_i)_{i \in V}$  are generated by the graph-directed iterated function system  $(V, E, (I_i)_{i \in V}, (S_e)_{e \in E})$  where each  $S_e : \mathbb{R} \rightarrow \mathbb{R}$  is a similarity map whose contracting ratio is denoted by  $r_e$  and assume further that the Open Set Condition is satisfied (the precise definitions of these concepts will be given in Section 1.1). For an edge  $e \in E$ , write  $i(e)$  for the initial vertex of  $e$  and we write  $t(e)$  for the terminal vertex of  $e$ . The averages in (1.1) and (1.2) are now obtained from (1.3) as follows. If we let  $\mathbf{p} = \mathbf{q} = (\frac{r_e}{\sum_{f \in E, i(f)=i(e)} r_f})_{e \in E}$ , then

(1.3) simplifies to (1.1), and if we let  $\mathbf{p} = \mathbf{q} = (u_{i(e)}^{-1} r_e^s u_{t(e)})_{e \in E}$  where  $s$  denotes the common Hausdorff dimension of the sets  $K_i$  and  $\mathbf{u} = (u_i)_{i \in V}$  is the unique positive normalised eigen-vector of the matrix  $(\sum_{e \in E, i(e)=i, t(e)=j} r_e^s)_{i,j \in V}$  with eigen-value equal to 1, then (1.3) simplifies to (1.2).

As an application of our main results, we compute the average distance between two points belonging to the Drobot-Turner set  $T_N(c, m)$  with respect to the normalised Hausdorff measure, i.e. we compute

$$\frac{1}{\mathcal{H}^s(T_N(c, m))^2} \int_{T_N(c, m)^2} |x - y| d(\mathcal{H}^s \times \mathcal{H}^s)(x, y),$$

where  $s$  denotes the Hausdorff dimension of  $T_N(c, m)$ , see Section 2.3 (and, in particular, (2.26) and (2.27)). Here the Drobot-Turner set is defined as follows, namely, for positive integers  $N$  and  $m$  and a positive real number  $c$ , the Drobot-Turner set  $T_N(c, m)$  is the set of those real numbers  $x \in [0, 1]$  for which any  $m$  consecutive base  $N$  digits in the  $N$ -ary expansion of  $x$  sum up to at least  $c$ . The set

$T_N(c, m)$  was introduced by Drobot & Turner [DrTu] in 1989. The importance of the Drobot-Turner set is not only due to its natural number theoretical nature but also because of the instrumental role that it has played for the past 20 years during the development of the theory of graph-directed constructions, see, for example, [MaWi] and the discussion in Cajar's text [Ca]. Because of the Drobot-Turner set's intricate and complicated nature, it is (perhaps) somewhat surprising that it is possible to find an explicit formula for the average distance given by  $\frac{1}{\mathcal{H}^s(T_N(c, m))^2} \int_{T_N(c, m)^2} |x - y| d(\mathcal{H}^s \times \mathcal{H}^s)(x, y)$ .

Our methods also have further applications. For example, they allow us to compute the first moment of the graph-directed self-similar measures  $\mu_{\mathbf{p}, i}$ . Recall, that the first moment  $M(\mu_{\mathbf{p}, i})$  of the measure  $\mu_{\mathbf{p}, i}$  is defined by

$$M(\mu_{\mathbf{p}, i}) = \int_{K_i} t d\mu_{\mathbf{p}, i}(t).$$

Using the results developed for computing the average distance in (1.3), we obtain explicit formulas for the moments  $M(\mu_{\mathbf{p}, i})$  in Theorem 2.6. Several corollaries follow easily from Theorem 2.6. For example, if  $s$  denotes the Hausdorff dimension of  $K_i$ , then an explicit formula for first moment

$$\frac{1}{\mathcal{H}^s(K_i)} \int_{K_i} t d\mathcal{H}^s(t).$$

of the normalised Hausdorff measure on  $K_i$  can be found using Theorem 2.6; this is the content of Corollary 2.7. Indeed, if we let  $\mathbf{p} = (u_{i(\mathbf{e})}^{-1} r_{\mathbf{e}}^s u_{t(\mathbf{e})})_{\mathbf{e} \in \mathbf{E}}$  where  $s$  denotes the common Hausdorff dimension of the sets  $K_i$  and  $\mathbf{u} = (u_i)_{i \in \mathbf{V}}$  is the unique positive normalised eigen-vector of the matrix  $(\sum_{\mathbf{e} \in \mathbf{E}, i(\mathbf{e})=i, t(\mathbf{e})=j} r_{\mathbf{e}}^s)_{i, j \in \mathbf{V}}$  with eigen-value equal to 1 (recall, that for an edge  $\mathbf{e} \in \mathbf{E}$ , write  $i(\mathbf{e})$  for the initial vertex of  $\mathbf{e}$  and we write  $t(\mathbf{e})$  for the terminal vertex of  $\mathbf{e}$ ), then  $M(\mu_{\mathbf{p}, i})$  equals  $\frac{1}{\mathcal{H}^s(K_i)} \int_{K_i} t d\mathcal{H}^s(t)$ ; the reader is referred to Corollary 2.7 for the details of this argument.

While this paper studies the average distance between points belonging to graph-directed self-similar subsets of  $\mathbb{R}$ , we note that the average distance between two points belonging to (non graph-directed) self-similar subsets of  $\mathbb{R}$  has recently been investigated by Leary et al [LeRuHa], Bailey et al [BaBoCrRo] and Allen et al [AlEdHaOl]. In particular, Leray et al found a formula for the limiting “geometric” average distance in (1.1) for self-similar subsets of  $\mathbb{R}$ . Averages similar to (1.1) between points belonging to self-similar subsets of  $\mathbb{R}$ , have also been studied in Bailey et al [BaBoCrRo]. In particular, Bailey et al are interested in developing numerical methods that allow for high-precision approximation of the integrals in (1.1). Following the work of Leary et al [LeRuHa] and Bailey et al [BaBoCrRo], Allen et al [AlEdHaOl] studied the more general averages in (1.3) for self-similar subsets of  $\mathbb{R}$ . Finally, we note that other notions of average distances on fractals (different from the ones considered in this paper and in [AlEdHaOl, BaBoCrRo, LeRuHa]) have been studied by Bandt & Kuschel [BaKu] and Hinz & Schief [HiSc].

**1.1 Graph-directed self-similar sets and graph-directed self-similar measures.** We now recall the definitions of graph-directed self-similar sets and measures, and introduce various notation that will be used throughout the paper. Fix a finite directed multigraph  $G = (\mathbf{V}, \mathbf{E})$  where  $\mathbf{V}$  denotes the set of vertices of  $G$  and  $\mathbf{E}$  denotes the set of edges of  $G$ . We will always assume that  $G$  is strongly connected, i.e. any two vertices are connected by a directed path of edges belonging to  $\mathbf{E}$ . For an edge  $\mathbf{e} \in \mathbf{E}$ , we write  $i(\mathbf{e})$  for the initial vertex of  $\mathbf{e}$  and we write  $t(\mathbf{e})$  for the terminal vertex of  $\mathbf{e}$ . For  $i, j \in \mathbf{V}$ , write

$$\begin{aligned} E_i &= \left\{ \mathbf{e} \in \mathbf{E} \mid i(\mathbf{e}) = i \right\}, \\ E_{i,j} &= \left\{ \mathbf{e} \in \mathbf{E} \mid i(\mathbf{e}) = i \text{ and } t(\mathbf{e}) = j \right\}; \end{aligned}$$

i.e.  $E_i$  is the family of all edges starting at  $i$ ; and  $E_{i,j}$  is the family of all edges starting at  $i$  and ending at  $j$ . Also, for a positive integer  $n$ , we write

$$\begin{aligned} \Sigma_G^n &= \left\{ e_1 \dots e_n \mid e_i \in \mathbf{E} \text{ for } 1 \leq i \leq n, \right. \\ &\quad t(e_1) = i(e_2), \\ &\quad t(e_{i-1}) = i(e_i) \text{ and } t(e_i) = i(e_{i+1}) \text{ for } 1 < i < n, \\ &\quad \left. t(e_{n-1}) = i(e_n) \right\}, \end{aligned}$$

i.e.  $\Sigma_G^n$  is the family of all finite strings  $\mathbf{i} = e_1 \dots e_n$  consisting of finite paths in  $G$  of length  $n$ . For a finite string  $\mathbf{i} = e_1 \dots e_n \in \Sigma_G^n$ , we write

$$i(\mathbf{i}) = i(e_1), \quad t(\mathbf{i}) = t(e_n).$$

Next, we recall the definition of graph-directed self-similar sets and measures. Since we are interested in graph-directed self-similar subsets of  $\mathbb{R}$ , we will only formulate the definition in this particular setting. A graph-directed self-similar iterated function system in  $\mathbb{R}$  is a list  $(V, E, (I_i)_{i \in V}, (S_e)_{e \in E})$  where:

- For each  $i \in V$  we have:  $I_i$  is a compact subinterval of  $\mathbb{R}$ .
- For each  $i, j \in V$  and  $e \in E_{i,j}$  we have:  $S_e : I_j \rightarrow I_i$  is a contractive similarity map, i.e. there are constants  $r_e$  and  $a_e$  with  $0 < r_e < 1$  and  $a_e \in \mathbb{R}$  such that for all  $x$ , we have

$$S_e(x) = r_e x + a_e.$$

It follows from [Fa,Hu] that there exists a unique list  $(K_i)_{i \in V}$  of non-empty compact sets  $K_i \subseteq I_i$  such that

$$K_i = \bigcup_{e \in E_i} S_e K_{t(e)}. \quad (1.4)$$

The sets  $(K_i)_{i \in V}$  are called the graph-directed self-similar sets associated with the list  $(V, E, (I_i)_{i \in V}, (S_e)_{e \in E})$ . The sets  $(K_i)_{i \in V}$  can also be constructed as follows. For  $\mathbf{i} = e_1 \dots e_n \in \Sigma_G^n$ , we write

$$\begin{aligned} r_{\mathbf{i}} &= r_{e_1} \cdots r_{e_n}, \\ S_{\mathbf{i}} &= S_{e_1} \cdots S_{e_n}, \\ I_{\mathbf{i}} &= S_{e_1} \cdots S_{e_n}(I_{t(e_n)}), \end{aligned} \quad (1.5)$$

and for a positive integer  $n$  and  $i \in V$ , let

$$K_{n,i} = \bigcup_{\substack{\mathbf{i} \in \Sigma_G^n \\ i(\mathbf{i})=i}} I_{\mathbf{i}}. \quad (1.6)$$

Then  $K_{0,i} \supseteq K_{1,i} \supseteq K_{2,i} \supseteq \dots$  and  $K_i$  equals the intersection of the  $K_{n,i}$ 's, i.e.

$$K_i = \bigcap_n K_{n,i}. \quad (1.7)$$

Loosely speaking (1.7) says that the sets  $K_{n,i}$  may be thought of as approximations to the set  $K_i$ ; this interpretation will be useful in Section 1.3. Assume in addition, we are given a family  $\mathbf{p} = (p_e)_{e \in E}$  where:

- For each  $i \in V$  we have:  $(p_e)_{e \in E_i}$  is a probability vector.

It follows from [Fa,Hu] that there exists a unique list  $(\mu_{\mathbf{p},i})_{i \in V}$  of probability measures with  $\text{supp } \mu_{\mathbf{p},i} = K_i$  such that

$$\mu_{\mathbf{p},i} = \sum_{e \in E_i} p_e \mu_{\mathbf{p},t(e)} \circ S_e^{-1}. \quad (1.8)$$

The measures  $(\mu_{\mathbf{p},i})_{i \in V}$  are called the graph-direct self-similar measures associated with the list  $(V, E, (I_i)_{i \in V}, (S_e)_{e \in E}, \mathbf{p})$ .

We will always assume that the Open Set condition (OSC) is satisfied. In the present setting, the OSC says that if  $i \in V$  and  $e, f \in E_i$  with  $e \neq f$ , then we have

$$S_e(I_{t(e)})^\circ \cap S_f(I_{t(f)})^\circ = \emptyset. \quad (1.9)$$

**1.2. Average distances: the measure theoretic approach.** For two Borel probability measures  $\mu$  and  $\nu$  on  $K_i$ , we define the average distance with respect to the measures  $\mu$  and  $\nu$  by

$$A_i(\mu, \nu) = \int_{K_i^2} |x - y| d(\mu \times \nu)(x, y). \quad (1.10)$$

**1.3. Average distances: the geometric approach.** There is a (perhaps) more intuitive approach for defining the average distance between two points belonging to  $K_i$ . This approach is described as follows. For Borel probability measures  $\mu$  and  $\nu$  on  $K_i$  and a positive integer  $n$ , we define the  $n$ 'th approximative average distance with respect to  $\mu$  and  $\nu$  by

$$A_{n,i}^{\text{geo}}(\mu, \nu) = \sum_{\substack{\mathbf{i}, \mathbf{j} \in \Sigma_G^n \\ i(\mathbf{i})=i(\mathbf{j})=i}} \frac{\mu(I_{\mathbf{i}})\nu(I_{\mathbf{j}})}{r_{\mathbf{i}}r_{\mathbf{j}}} \int_{I_{\mathbf{i}} \times I_{\mathbf{j}}} |x - y| d(x, y). \quad (1.11)$$

Finally, we define the geometric average distance with respect to  $\mu$  and  $\nu$  by

$$A_i^{\text{geo}}(\mu, \nu) = \lim_n A_{n,i}^{\text{geo}}(\mu, \nu), \quad (1.12)$$

provided the limit exists.

The number  $A_{n,i}^{\text{geo}}(\mu, \nu)$  has a clear geometric interpretation. Namely, two players A and B, say, throw darts at the  $n$ 'th approximation  $K_{n,i} = \cup_{\mathbf{i} \in \Sigma_G^n, i(\mathbf{i})=i} I_{\mathbf{i}}$  to the set  $K_i$ . For each  $\mathbf{i} \in \Sigma_G^n$  with  $i(\mathbf{i}) = i$ , player A has the probability  $\mu(I_{\mathbf{i}})$  of hitting  $I_{\mathbf{i}}$ , and player B has the probability  $\nu(I_{\mathbf{i}})$  of hitting  $I_{\mathbf{i}}$ . The number  $A_{n,i}^{\text{geo}}(\mu, \nu)$  is now the average distance between a dart thrown by A and a dart thrown by B.

**1.4. Linking the measure theoretic approach and the geometric approach.** The next result shows that the geometric approach in (1.11) and (1.12) leads to the same notion of average distance as the measure theoretical approach in (1.10); more precisely, the result shows that the limit  $A_i^{\text{geo}}(\mu, \nu) = \lim_n A_{n,i}^{\text{geo}}(\mu, \nu)$  always exists and equals  $A_i(\mu, \nu)$

**Proposition 1.1.** *Let  $\mu$  and  $\nu$  be non-atomic Borel probability measures on  $K_i$ . Then the limit  $A_i^{\text{geo}}(\mu, \nu) = \lim_n A_{n,i}^{\text{geo}}(\mu, \nu)$  exists and*

$$A_i^{\text{geo}}(\mu, \nu) = A_i(\mu, \nu).$$

*Proof.*

For  $\mathbf{i} \in \Sigma_G^n$ , let  $\lambda_{\mathbf{i}}$  denote the normalized Lebesgue measure restricted to  $I_{\mathbf{i}}$ . Next, for a positive integer  $n$ , define measures  $\tilde{\mu}_n$  and  $\tilde{\nu}_n$  by  $\tilde{\mu}_n = \sum_{\mathbf{i} \in \Sigma_G^n, i(\mathbf{i})=i} \mu(I_{\mathbf{i}}) \lambda_{\mathbf{i}}$  and  $\tilde{\nu}_n = \sum_{\mathbf{i} \in \Sigma_G^n, i(\mathbf{i})=i} \nu(I_{\mathbf{i}}) \lambda_{\mathbf{i}}$ . Since  $\mu$  and  $\nu$  are non-atomic, it is not difficult to see that  $\tilde{\mu}_n \rightarrow \mu$  weakly and that  $\tilde{\nu}_n \rightarrow \nu$  weakly, and it therefore follows from [Bi, Section 3.4] that  $\tilde{\mu}_n \times \tilde{\nu}_n \rightarrow \mu \times \nu$ . In particular, since clearly  $A_{n,i}^{\text{geo}}(\mu, \nu) = \int |x - y| d(\tilde{\mu}_n \times \tilde{\nu}_n)(x, y)$  and  $A_i(\mu, \nu) = \int |x - y| d(\mu \times \nu)(x, y)$ , this now implies that

$$A_{n,i}^{\text{geo}}(\mu, \nu) = \int |x - y| d(\tilde{\mu}_n \times \tilde{\nu}_n)(x, y) \rightarrow \int |x - y| d(\mu \times \nu)(x, y) = A_i(\mu, \nu).$$

This completes the proof.  $\square$

## 2. STATEMENTS OF RESULTS.

We will now state our main results on average distances and moments of graph-directed of self-similar measures. We therefore fix self-similar measures  $\mu_{\mathbf{p},i}$  and  $\mu_{\mathbf{q},i}$  associated with two (not necessarily identical) families  $\mathbf{p} = (p_e)_{e \in E}$  and  $\mathbf{q} = (q_e)_{e \in E}$  of positive numbers  $p_e$  and  $q_e$  such that  $(p_e)_{e \in E_i}$  and  $(q_e)_{e \in E_i}$  are probability vectors for all  $i \in V$ ; hence, the measures  $(\mu_{\mathbf{p},i})_{i \in V}$  and  $(\mu_{\mathbf{q},i})_{i \in V}$  are the unique Borel probability measures satisfying

$$\begin{aligned}\mu_{\mathbf{p},i} &= \sum_{e \in E_i} p_e \mu_{\mathbf{p},t(e)} \circ S_e^{-1}, \\ \mu_{\mathbf{q},i} &= \sum_{e \in E_i} q_e \mu_{\mathbf{q},t(e)} \circ S_e^{-1},\end{aligned}\tag{2.1}$$

for all  $i \in V$ . In Section 2.1 we present our results on average distances and in Section 2.2 we present our results on moments of graph-directed self-similar measures. Finally, in Section 2.3 we illustrate our results by computing the average distance between two belonging to the Drobot-Turner set.

**2.1. Average distances.** We first compute the average distances  $A_i(\mu_{\mathbf{p},i}, \mu_{\mathbf{q},i})$  and  $A_i^{\text{geo}}(\mu_{\mathbf{p},i}, \mu_{\mathbf{q},i})$  with respect to the two graph-directed self-similar measures  $\mu_{\mathbf{p},i}$  and  $\mu_{\mathbf{q},i}$  in (2.1). This result is the content of the next theorem.

**Theorem 2.1.** *Let  $\mathbf{p} = (p_e)_{e \in E}$  and  $\mathbf{q} = (q_e)_{e \in E}$  be families of positive numbers  $p_e$  and  $q_e$  such that  $(p_e)_{e \in E_i}$  and  $(q_e)_{e \in E_i}$  are probability vectors for all  $i \in V$ , and let  $(\mu_{\mathbf{p},i})_{i \in V}$  and  $(\mu_{\mathbf{q},i})_{i \in V}$  be the associated graph-directed self-similar measures, i.e.  $(\mu_{\mathbf{p},i})_{i \in V}$  and  $(\mu_{\mathbf{q},i})_{i \in V}$  are the unique measures satisfying (2.1).*

*Define the matrices  $S_{\mathbf{p}}$  and  $S_{\mathbf{q}}$  and the vectors  $T_{\mathbf{p}}$  and  $T_{\mathbf{q}}$  by*

$$\begin{aligned}S_{\mathbf{p}} &= \left( \sum_{e \in E_{i,j}} p_e r_e \right)_{i,j \in V}, \quad S_{\mathbf{q}} = \left( \sum_{e \in E_{i,j}} q_e r_e \right)_{i,j \in V}, \\ T_{\mathbf{p}} &= \left( \sum_{e \in E_i} p_e a_e \right)_{i \in V}, \quad T_{\mathbf{q}} = \left( \sum_{e \in E_i} q_e a_e \right)_{i \in V}.\end{aligned}$$

*Then  $1 - S_{\mathbf{p}}$  and  $1 - S_{\mathbf{q}}$  are invertible, and we can define the matrices  $(B_{\mathbf{p},i})_{i \in V}$  and  $(B_{\mathbf{q},i})_{i \in V}$  by*

$$(B_{\mathbf{p},i})_{i \in V} = (1 - S_{\mathbf{p}})^{-1} T_{\mathbf{p}}, \quad (B_{\mathbf{q},i})_{i \in V} = (1 - S_{\mathbf{q}})^{-1} T_{\mathbf{q}}.$$

*Define the vector  $Y$  by*

$$Y = \left( \sum_{e,f \in E_i} p_e q_f \left| (r_e B_{\mathbf{p},t(e)} + a_e) - (r_f B_{\mathbf{q},t(f)} + a_f) \right| \right)_{i \in V},$$

*and define the matrix  $M$  by*

$$M = \left( \sum_{e \in E_{i,j}} p_e q_e r_e \right)_{i,j \in V}.$$

*Then  $1 - M$  is invertible and*

$$(A_i(\mu_{\mathbf{p},i}, \mu_{\mathbf{q},i}))_{i \in V} = (A_i^{\text{geo}}(\mu_{\mathbf{p},i}, \mu_{\mathbf{q},i}))_{i \in V} = (1 - M)^{-1} Y.$$

The proof of Theorem 2.1 is given in Section 3.

If  $\mathbf{p} = \mathbf{q}$  and all the contraction ratios  $r_e$  coincide, then the formula in Theorem 2.1 for the average  $A_i(\mu_{\mathbf{p},i}, \mu_{\mathbf{q},i})$  simplifies; this is the context of the next corollary.

**Corollary 2.2.** *Let  $\mathbf{p} = (p_e)_{e \in E}$  be a family of positive numbers  $p_e$  such that  $(p_e)_{e \in E_i}$  is a probability vector for all  $i \in V$ , and let  $(\mu_{\mathbf{p},i})_{i \in V}$  be the associated graph-directed self-similar measures, i.e.  $(\mu_{\mathbf{p},i})_{i \in V}$  are the unique measures satisfying (2.1). Assume that there is a real number  $r$  such that  $r_e = r$  for all  $e \in E$ .*

*Define the matrix  $S$  and the vectors  $T$  by*

$$S = r \left( \sum_{e \in E_{i,j}} p_e \right)_{i,j \in V},$$

$$T = \left( \sum_{e \in E_i} p_e a_e \right)_{i \in V}.$$

*Then  $1 - S$  is invertible, and we can define the matrix  $(B_i)_{i \in V}$  by*

$$(B_i)_{i \in V} = (1 - S)^{-1} T.$$

*Define the vector  $Y$  by*

$$Y = \left( \sum_{e,f \in E_i} p_e p_f \left| (r B_{t(e)} + a_e) - (r B_{t(f)} + a_f) \right| \right)_{i \in V},$$

*and define the matrix  $M$  by*

$$M = r \left( \sum_{e \in E_{i,j}} p_e^2 \right)_{i,j \in V}.$$

*Then  $1 - M$  is invertible and*

$$(A_i(\mu_{\mathbf{p},i}, \mu_{\mathbf{p},i}))_{i \in V} = (A_i^{\text{geo}}(\mu_{\mathbf{p},i}, \mu_{\mathbf{p},i}))_{i \in V} = (1 - M)^{-1} Y.$$

*Proof.*

This follows immediately from Theorem 2.1. □

Below we consider two further corollaries of Theorem 2.1. By applying Theorem 2.1 to the families  $\mathbf{p} = \mathbf{q} = \mathbf{v}$  where  $\mathbf{v} = (\frac{r_e}{R_i(e)})_{e \in E}$  and  $R_i = \sum_{e \in E_i} r_e$  for  $i \in V$ , we obtain the first corollary. This corollary shows that the following natural limiting geometric average distance exists, namely,

$$\lim_n \frac{\int_{K_{n,i}^2} |x-y| d(x,y)}{\int_{K_{n,i}^2} d(x,y)}, \text{ and provides an explicit value for it.}$$

**Corollary 2.3.** *For  $i \in V$ , write  $R_i = \sum_{e \in E_i} r_e$ , and define the matrix  $S$  and the vector  $T$  by*

$$S = \left( \frac{1}{R_i} \sum_{e \in E_{i,j}} r_e^2 \right)_{i,j \in V},$$

$$T = \left( \frac{1}{R_i} \sum_{e \in E_i} r_e a_e \right)_{i \in V}.$$

*Then  $1 - S$  is invertible, and we can define the matrix  $(B_i)_{i \in V}$  by*

$$(B_i)_{i \in V} = (1 - S)^{-1} T.$$

*Define the vector  $Y$  by*

$$Y = \left( \frac{1}{R_i^2} \sum_{e,f \in E_i} r_e r_f \left| (r_e B_{t(e)} + a_e) - (r_f B_{t(f)} + a_f) \right| \right)_{i \in V},$$

*and define the matrix  $M$  by*

$$M = \left( \frac{1}{R_i^2} \sum_{e \in E_{i,j}} r_e^3 \right)_{i,j \in V}.$$

*Then  $1 - M$  is invertible and*

$$\left( \lim_n \frac{\int_{K_{n,i}^2} |x-y| d(x,y)}{\int_{K_{n,i}^2} d(x,y)} \right)_{i \in V} = (1 - M)^{-1} Y. \quad (2.2)$$

*Proof.*

For  $e \in E$ , write  $v_e = \frac{r_e}{R_i(e)}$ , and put  $\mathbf{v} = (v_e)_{e \in E}$ . It is clear that  $(v_e)_{e \in E_i}$  is a probability vector for each  $i \in V$ , and the graph-directed self-similar measures  $(\mu_{\mathbf{v},i})_{i \in V}$  associated with  $\mathbf{v}$  are therefore well-defined. It is also clear that

$$A_{n,i}^{\text{geo}}(\mu_{\mathbf{v},i}, \mu_{\mathbf{v},i}) = \frac{\int_{K_{n,i}^2} |x-y| d(x,y)}{\int_{K_{n,i}^2} d(x,y)},$$

and the result therefore follows immediately from Theorem 2.1.  $\square$



The next corollary, i.e. Corollary 2.4, computes the average distance between two points in  $K_i$  with respect to the natural uniform distribution on  $K_i$ , namely, the normalised Hausdorff measure. To state this result, we introduce the following notation and terminology. If  $A$  is a square matrix, then we will denote the spectral radius of  $A$  by  $\text{spec-rad } A$ . Also, for a positive number  $t$ , let  $\mathcal{H}^t$  denote the  $t$ -dimensional Hausdorff measure. Corollary 2.4 now gives an explicit value for the average distance between two points in  $K_i$  with respect to the normalised Hausdorff measure, i.e.  $\frac{1}{\mathcal{H}^s(K_i)^2} \int_{K_i^2} |x - y| d(\mathcal{H}^s \times \mathcal{H}^s)(x, y)$  where  $s$  denotes the Hausdorff dimension of  $K_i$ .

**Corollary 2.4.** *Let  $s$  denote the Hausdorff dimension of  $K_i$ , i.e.  $s$  is the unique real number determined by the following: if the matrix  $\Delta_t$  is defined by*

$$\Delta_t = \left( \sum_{e \in E_{i,j}} r_e^t \right)_{i,j \in V}, \quad (2.3)$$

for  $t \in \mathbb{R}$ , then  $s$  is the unique real number such that

$$\text{spec-rad } \Delta_s = 1,$$

see [MaWi]. Let  $\mathbf{u} = (u_i)_{i \in V}$  be the unique vector with  $u_i > 0$  for all  $i$  and  $\sum_i u_i = 1$  such that

$$\Delta_s \mathbf{u} = \mathbf{u}, \quad (2.4)$$

and define the matrix  $S$  and the vector  $T$  by

$$S = \left( \sum_{e \in E_{i,j}} u_i^{-1} r_e^{s+1} u_j \right)_{i,j \in V},$$

$$T = \left( \sum_{e \in E_i} u_i^{-1} r_e^s a_e u_{t(e)} \right)_{i \in V}.$$

Then  $1 - S$  is invertible, and we can define the matrix  $(B_i)_{i \in V}$  by

$$(B_i)_{i \in V} = (1 - S)^{-1} T.$$

Define the vector  $Y$  by

$$Y = \left( \sum_{e,f \in E_i} u_i^{-2} r_e^s r_f^s u_{t(e)} u_{t(f)} \left| (r_e B_{t(e)} + a_e) - (r_f B_{t(f)} + a_f) \right| \right)_{i \in V},$$

and define the matrix  $M$  by

$$M = \left( \sum_{e \in E_{i,j}} u_i^{-2} r_e^{2s+1} u_j^2 \right)_{i,j \in V}.$$

Then  $1 - M$  is invertible and

$$\left( \frac{1}{\mathcal{H}^s(K_i)^2} \int_{K_i^2} |x - y| d(\mathcal{H}^s \times \mathcal{H}^s)(x, y) \right)_{i \in V} = (1 - M)^{-1} Y. \quad (2.5)$$

*Proof.*

For  $e \in E$ , write  $h_e = u_{i(e)}^{-1} r_e^s u_{t(e)}$ , and put  $\mathbf{h} = (h_e)_{e \in E}$ . Since  $\Delta_s \mathbf{u} = \mathbf{u}$ , we conclude that  $(h_e)_{e \in E_i}$  is a probability vector for each  $i \in V$ , and the graph-directed self-similar measures  $(\mu_{\mathbf{h},i})_{i \in V}$  associated with  $\mathbf{h}$  are therefore well-defined. It is well-known that the measure  $\mu_{\mathbf{h},i}$  equals the normalised  $s$ -dimensional Hausdorff measure restricted to  $K_i$  (see, for example, [Sp]), whence

$$A_i(\mu_{\mathbf{h},i}, \mu_{\mathbf{h},i}) = \frac{1}{\mathcal{H}^s(K_i)^2} \int_{K_i^2} |x - y| d(\mathcal{H}^s \times \mathcal{H}^s)(x, y),$$

and the result therefore follows immediately from Theorem 2.1.  $\square$

If the graph-directed iterated function system  $(V, E, (I_i)_{i \in V}, (S_e)_{e \in E})$  exhibits some degree of uniformity, then one would expect that the “natural” averages in (2.2) and (2.5) coincide. Below we will state and prove a precise version of this statement. In particular, we will prove that if all the contraction ratios  $r_e$  coincide (i.e. if there is a number  $r$  such that  $r_e = r$  for all  $e$ ) and all vertices have the same outdegree (i.e. there is positive integer  $N$  such that  $|E_i| = N$  for all  $i$ ), then the two “natural” averages in (2.2) and (2.5) coincide.

**Corollary 2.5.** *Let  $s$  denote the Hausdorff dimension of  $K_i$ . Assume that there are real numbers  $r$  and  $N$  such that  $r_e = r$  for all  $e$  and  $|E_i| = N$  for all  $i$ . Define the vector  $Y$  by*

$$Y = \frac{1}{N^2} \left( \sum_{e, f \in E_i} |a_e - a_f| \right)_{i \in V},$$

and define the matrix  $M$  by

$$M = \frac{r}{N^2} \left( |E_{i,j}| \right)_{i,j \in V}.$$

Then  $1 - M$  is invertible and

$$\left( \lim_n \frac{\int_{K_{n,i}^2} |x - y| d(x, y)}{\int_{K_{n,i}^2} d(x, y)} \right)_{i \in V} = \left( \frac{1}{\mathcal{H}^s(K_i)^2} \int_{K_i^2} |x - y| d(\mathcal{H}^s \times \mathcal{H}^s)(x, y) \right)_{i \in V} = (1 - M)^{-1} Y.$$

*Proof.*

We first prove that

$$\left( \lim_n \frac{\int_{K_{n,i}^2} |x - y| d(x, y)}{\int_{K_{n,i}^2} d(x, y)} \right)_{i \in V} = (1 - M)^{-1} Y. \quad (2.6)$$

Indeed, it is clear that  $R_i = \sum_{e \in E_i} r_e |E_i| = rN$  for all  $i$ , and (2.6) therefore follows immediately from Corollary 2.3.

Next, we prove that

$$\left( \frac{1}{\mathcal{H}^s(K_i)^2} \int_{K_i^2} |x - y| d(\mathcal{H}^s \times \mathcal{H}^s)(x, y) \right)_{i \in V} = (1 - M)^{-1} Y. \quad (2.7)$$

The proof of (2.7) is based on the following part of the Perron-Frobenius theorem. In order to state this result, we recall the following terminology and notation, namely, a square  $k$  times  $k$  matrix  $A$  with real entries is called irreducible if for all  $i, j = 1, \dots, k$  there exists a positive integer  $n$  such that  $(A^n)_{ij} > 0$ . If  $A = (a_{i,j})_{1 \leq i \leq k, 1 \leq j \leq k}$  is a matrix with real entries, then we will write  $A \geq 0$  if  $a_{i,j} \geq 0$  for all  $i, j$ . We can now state the part of the Perron-Frobenius theorem that is needed in the proof of (2.7); a proof of Perron-Frobenius theorem can be found in [Ga, Chapter XIII].

*(Part of the) Perron-Frobenius Theorem.* *Let  $A \geq 0$  be an irreducible square matrix and let  $\lambda \in \mathbb{R}$ . If there exists a non-zero column vector  $\mathbf{x}$  with  $\mathbf{x} \geq 0$  such that  $A\mathbf{x} = \lambda\mathbf{x}$ , then  $\lambda = \text{spec-rad } A$ .*

Let  $\Delta_s$  be the matrix defined in (2.3) and note that  $\Delta_s$  is irreducible (because the graph  $G$  is assumed to be strongly connected). Next, let  $\mathbf{x} = (x_i)_{i \in V} = (\frac{1}{|V|})_{i \in V}$  and observe that

$$\begin{aligned} \Delta_s \mathbf{x} &= \left( \sum_{j \in V} \left( \sum_{e \in E_{i,j}} r_e^s \right) \frac{1}{|V|} \right)_{i \in V} \\ &= \left( \frac{1}{|V|} r^s \sum_{j \in V} |E_{i,j}| \right)_{i \in V} \\ &= \left( \frac{1}{|V|} r^s |E_i| \right)_{i \in V} \\ &= \left( \frac{1}{|V|} r^s N \right)_{i \in V} \\ &= r^s N \mathbf{x}, \end{aligned} \quad (2.8)$$

where we have used the fact that  $\sum_{j \in V} |E_{ij}| = |E_i|$ . Since  $\mathbf{x} \geq 0$ , it follows from (2.8) and Perron-Frobenius theorem that  $r^s N = \text{spec-rad } \Delta_s = 1$ .

Next observe that  $r_e^s = r^s = \frac{1}{N}$  for all  $e$ ; indeed this follows immediately from the equation  $r^s N = 1$ .

Also observe that if  $\mathbf{u} = (u_i)_{i \in V}$  is the unique vector with  $\mathbf{u} \geq 0$  and  $\sum_i u_i = 1$  satisfying (2.4), then  $u_i = \frac{1}{|V|}$  for all  $i$ ; indeed, since  $\Delta_s \mathbf{x} = \mathbf{x}$  (by (2.8) because  $r^s N = 1$ ) and  $\sum_{i \in V} x_i = 1$ , we conclude from the uniqueness of the vector  $\mathbf{u} = (u_i)_{i \in V}$  that  $\mathbf{u} = \mathbf{x}$ , i.e.  $u_i = x_i = \frac{1}{|V|}$  for all  $i$ .

Summarising, we have  $r_e^s = r^s = \frac{1}{N}$  for all  $e$  and  $u_i = \frac{1}{|V|}$  for all  $i$ , and (2.7) follows from this and Corollary 2.4.  $\square$

**2.2. Moments.** We also compute the moment of  $\mu_{\mathbf{p},i}$ ; this is the content of Theorem 2.6 below. Recall, that if  $\mu$  is a probability measure on  $K_i$ , then the moment of  $\mu$  is defined by

$$M_i(\mu) = \int_{K_i} t d\mu(t).$$

**Theorem 2.6.** *Let  $\mathbf{p} = (p_e)_{e \in E}$  be a family of positive numbers  $p_e$  such that  $(p_e)_{e \in E_i}$  are probability vectors for all  $i \in V$ , and let  $(\mu_{\mathbf{p},i})_{i \in V}$  be the associated graph-directed self-similar measures, i.e.  $(\mu_{\mathbf{p},i})_{i \in V}$  are the unique measures satisfying (2.1).*

*Define the matrix  $S$  and the vector  $T$  by*

$$S = \left( \sum_{e \in E_{i,j}} p_e r_e \right)_{i,j \in V},$$

$$T = \left( \sum_{e \in E_i} p_e a_e \right)_{i \in V}.$$

*Then  $1 - S$  is invertible and*

$$(M_i(\mu_{\mathbf{p},i}))_{i \in V} = (1 - S)^{-1} T.$$

The proof of Theorem 2.6 is given in Section 4.

As a corollary to Theorem 2.6 we now compute the moment of the  $s$ -dimensional Hausdorff measure on  $K_i$  where  $s$  denotes the Hausdorff dimension of  $K_i$ . Recall that if  $t$  is a positive real number, then  $\mathcal{H}^t$  denotes the  $t$ -dimensional Hausdorff measure.

**Corollary 2.7.** *Let  $s$  denote the Hausdorff dimension of  $K_i$  and let  $\mathbf{u} = (u_i)_{i \in V}$  be the unique vector with  $u_i > 0$  for all  $i$  and  $\sum_i u_i = 1$  satisfying (2.4). Define the matrix  $S$  and the vector  $T$  by*

$$S = \left( \sum_{e \in E_{i,j}} u_i^{-1} r_e^{s+1} u_j \right)_{i,j \in V},$$

$$T = \left( \sum_{e \in E_i} u_i^{-1} r_e^s a_e u_{t(e)} \right)_{i \in V}.$$

*Then  $1 - S$  is invertible and*

$$\left( \frac{1}{\mathcal{H}^s(K_i)} \int_{K_i} t d\mathcal{H}^s(t) \right)_{i \in V} = (1 - S)^{-1} T.$$

*Proof.*

Let  $\mathbf{h} = (h_e)_{e \in E}$  be the family of positive real numbers  $h_e$  from the proof of Corollary 2.4, i.e.  $h_e = u_{i(e)}^{-1} r_e^s u_{t(e)}$ . It follows from the proof of Corollary 2.4 that  $(h_e)_{e \in E_i}$  is a probability vector for each  $i \in V$ ,

and the graph-directed self-similar measures  $(\mu_{\mathbf{h},i})_{i \in \mathbf{V}}$  associated with  $\mathbf{h}$  are therefore well-defined. It also follows from the proof of Corollary 2.4 that the measure  $\mu_{\mathbf{h},i}$  equals the normalised  $s$ -dimensional Hausdorff measure restricted to  $K_i$ , whence

$$M_i(\mu_{\mathbf{h},i}) = \frac{1}{\mathcal{H}^s(K_i)} \int_{K_i} t d\mathcal{H}^s(t),$$

and the result therefore follows immediately from Theorem 2.6.  $\square$

**2.3. Example: The average distance between points in the Drobot-Turner set.** As an example of our results, we will now compute the average distance between two points in the Drobot-Turner set with respect to the normalised Hausdorff measure. For positive integers  $N$  and  $m$  and a positive real number  $c$ , the Drobot-Turner set  $T_N(c, m)$  is defined as those real numbers  $x \in [0, 1]$  for which any  $m$  consecutive base  $N$  digits in the  $N$ -ary expansion of  $x$  sum up to at least  $c$ , see [DrTu]. For  $x \in [0, 1]$ , let  $x = \sum_{n=1}^{\infty} \frac{d_i(x)}{N^n}$ , with  $d_i(x) \in \{0, 1, \dots, N-1\}$ , be the unique non-terminating  $N$ -ary expansion of  $x$ . Using this notation, the set  $T_N(c, m)$  is given by

$$T_N(c, m) = \left\{ x \in [0, 1] \mid d_{i+1}(x) + \dots + d_{i+m}(x) \geq c \text{ for all } i \right\}.$$

We will now compute the average distance between two points in the set  $T_N(c, m)$  with respect to the Hausdorff measure on  $T_N(c, m)$ , i.e. if  $s$  denotes the Hausdorff dimension of  $T_N(c, m)$ , then we will compute the average given by

$$\frac{1}{\mathcal{H}^s(T_N(c, m))^2} \int_{T_N(c, m)^2} |x - y| d(\mathcal{H}^s \times \mathcal{H}^s)(x, y). \quad (2.9)$$

The key observation allowing us to compute the average in (2.9) is the following, namely: while the Drobot-Turner set  $T_N(c, m)$  is not a graph-directed self-similar set, it is, nevertheless, a finite union of graph-directed self-similar sets. This was first observed by Mauldin & Williams [MaWi]; we will now describe Mauldin & Williams' construction. For a positive integer  $n$ , write  $\Pi_n = \{0, 1, \dots, N-1\}^n$  for the family of all strings  $\mathbf{i} = i_1 \dots i_n$  of length  $n$  with entries  $i_j \in \{0, 1, \dots, N-1\}$ . Also, for  $\mathbf{i} = i_1 \dots i_n \in \Pi_n$ , write  $\mathbb{I}_{\mathbf{i}} = [s_{\mathbf{i}}, t_{\mathbf{i}}]$  where  $s_{\mathbf{i}} = \frac{i_1}{N} + \dots + \frac{i_n}{N^n}$  and  $t_{\mathbf{i}} = \frac{i_1}{N} + \dots + \frac{i_n}{N^n} + \frac{1}{N^n}$ . We now define the graph-directed iterated function system  $(\mathbf{V}, \mathbf{E}, (I_{\mathbf{i}})_{\mathbf{i} \in \mathbf{V}}, (S_{\mathbf{e}})_{\mathbf{e} \in \mathbf{E}})$  as follows. Let

$$\mathbf{V} = \left\{ i_1 \dots i_m \in \Pi_m \mid i_1 + \dots + i_m \geq c \right\} \quad (2.10)$$

and

$$\mathbf{E} = \left\{ (i_1 \dots i_m, j_1 \dots j_m) \in \mathbf{V} \times \mathbf{V} \mid j_1 \dots j_{m-1} = i_2 \dots i_m \right\}. \quad (2.11)$$

Next, for  $\mathbf{i} \in \mathbf{V}$ , let

$$I_{\mathbf{i}} = \mathbb{I}_{\mathbf{i}} \quad (2.12)$$

and for  $(\mathbf{i}, \mathbf{j}) = (i_1 \dots i_m, j_1 \dots j_m) \in \mathbf{E}$ , let  $S_{(\mathbf{i}, \mathbf{j})} : \mathbb{I}_{\mathbf{j}} \rightarrow \mathbb{I}_{\mathbf{i}}$  be the unique increasing affine map that maps  $\mathbb{I}_{\mathbf{j}}$  onto  $\mathbb{I}_{\mathbf{i}}$ , i.e.

$$S_{(\mathbf{i}, \mathbf{j})}(x) = \frac{1}{N}x + \frac{i_1}{N}; \quad (2.13)$$

in particular, it follows that  $r_{\mathbf{e}} = r_{(\mathbf{i}, \mathbf{j})} = \frac{1}{N}$  and  $a_{\mathbf{e}} = a_{(\mathbf{i}, \mathbf{j})} = \frac{i_1}{N}$  for  $\mathbf{e} = (\mathbf{i}, \mathbf{j}) = (i_1 \dots i_m, j_1 \dots j_m) \in \mathbf{E}$ . We now let  $(K_{\mathbf{i}})_{\mathbf{i} \in \mathbf{V}}$  be the graph-directed self-similar sets associated with the construction (2.10)–(2.13), i.e.  $(K_{\mathbf{i}})_{\mathbf{i} \in \mathbf{V}}$  is the unique family of non-empty compact sets satisfying  $K_{\mathbf{i}} = \bigcup_{(\mathbf{i}, \mathbf{j}) \in \mathbf{E}} S_{(\mathbf{i}, \mathbf{j})} K_{\mathbf{j}}$  for all  $\mathbf{i} \in \mathbf{V}$ . A moments reflection shows that  $\mathbb{I}_{\mathbf{i}} \cap T_N(c, m) = \bigcup_{(\mathbf{i}, \mathbf{j}) \in \mathbf{E}} S_{(\mathbf{i}, \mathbf{j})}(\mathbb{I}_{\mathbf{j}} \cap T_N(c, m))$  for all  $\mathbf{i} \in \mathbf{V}$ , and the uniqueness of the sets  $(K_{\mathbf{i}})_{\mathbf{i} \in \mathbf{V}}$  therefore implies that

$$\mathbb{I}_{\mathbf{i}} \cap T_N(c, m) = K_{\mathbf{i}}, \quad (2.14)$$

i.e. the part of the Drobot-Turner set that lies in  $\mathbb{I}_{\mathbf{i}}$  equals  $K_{\mathbf{i}}$ . Also, since it is not difficult to see that  $T_N(c, m) = \bigcup_{\mathbf{i} \in \mathbf{V}} (\mathbb{I}_{\mathbf{i}} \cap T_N(c, m))$ , we deduce from (2.14) that

$$T_N(c, m) = \bigcup_{\mathbf{i} \in \mathbf{V}} K_{\mathbf{i}}, \quad (2.15)$$

i.e. the Drobot-Turner set is the union of the  $K_{\mathbf{i}}$ 's. Finally, we note that while graph-directed self-similar sets, in general, are not pairwise disjoint, the sets  $K_{\mathbf{i}}$  nevertheless satisfy

$$\mathcal{H}^s(K_{\mathbf{i}} \cap K_{\mathbf{j}}) = 0 \quad (2.16)$$

for all  $\mathbf{i}$  and  $\mathbf{j}$  with  $\mathbf{i} \neq \mathbf{j}$  (because  $K_{\mathbf{i}} \subseteq \mathbb{I}_{\mathbf{i}}$  and the interiors of the intervals  $\mathbb{I}_{\mathbf{i}}$  are pairwise disjoint). Statements (2.14)–(2.16) together with Corollary 2.4 and Corollary 2.7 play a crucial role in the computation of the average in (2.9). In order to describe this, we introduce the following abbreviated notation. Namely, we write

$$A_{\mathbf{i}} = \frac{1}{\mathcal{H}^s((\mathbb{I}_{\mathbf{i}} \cap T_N(c, m))^2)} \int_{(\mathbb{I}_{\mathbf{i}} \cap T_N(c, m))^2} |x - y| d(\mathcal{H}^s \times \mathcal{H}^s)(x, y),$$

$$M_{\mathbf{i}} = \frac{1}{\mathcal{H}^s(\mathbb{I}_{\mathbf{i}} \cap T_N(c, m))} \int_{\mathbb{I}_{\mathbf{i}} \cap T_N(c, m)} t d\mathcal{H}^s(t),$$

for  $\mathbf{i} \in \mathbb{V}$ . The computation of (2.9) is divided into the following two parts:

*Part 1:* Using (2.15) and (2.16), we express the average in (2.9) in terms of the  $A_{\mathbf{i}}$ 's and the  $M_{\mathbf{i}}$ 's.

*Part 2:* Using (2.14) and Corollaries 2.4 and 2.7, we derive explicit expressions for  $A_{\mathbf{i}}$  and  $M_{\mathbf{i}}$ .

Part 1 and Part 2 will be proved below. However, we first introduce the following notation. Recall, that  $s$  denotes the Hausdorff dimension of  $T_N(c, m)$ . Since  $\mathbb{I}_{\mathbf{i}} \cap T_N(c, m) = K_{\mathbf{i}}$ , we conclude that  $s$  equals the Hausdorff dimension of  $K_{\mathbf{i}}$ , and it therefore follows from Corollary 2.4 that  $s$  is the unique real number determined by the following. For  $\mathbf{i}, \mathbf{j} \in \mathbb{V}$ , let

$$\delta_{\mathbf{i}, \mathbf{j}} = \begin{cases} 0 & \text{if } E_{\mathbf{i}, \mathbf{j}} = \emptyset; \\ 1 & \text{if } E_{\mathbf{i}, \mathbf{j}} \neq \emptyset \end{cases}$$

(in other words,  $\delta_{\mathbf{i}, \mathbf{j}} = |E_{\mathbf{i}, \mathbf{j}}|$ ), and put

$$\Delta = (\delta_{\mathbf{i}, \mathbf{j}})_{\mathbf{i}, \mathbf{j} \in \mathbb{V}},$$

$$\rho = \text{spec-rad } \Delta.$$

If the matrix  $\Delta_t$  is defined by  $\Delta_t = (\sum_{e \in E_{\mathbf{i}, \mathbf{j}}} r_e^t)_{\mathbf{i}, \mathbf{j} \in \mathbb{V}} = \frac{1}{N^t} (|E_{\mathbf{i}, \mathbf{j}}|)_{\mathbf{i}, \mathbf{j} \in \mathbb{V}} = \frac{1}{N^t} \Delta$ , for  $t \in \mathbb{R}$ , then  $s$  is the unique real number such that  $\text{spec-rad } \Delta_s = 1$ . We conclude from this that  $s$  is the unique real number such that

$$\frac{1}{N^s} \rho = 1. \quad (2.17)$$

Next, let  $\mathbf{u} = (u_{\mathbf{i}})_{\mathbf{i} \in \mathbb{V}}$  be the unique vector with  $u_{\mathbf{i}} > 0$  for all  $\mathbf{i}$  and  $\sum_{\mathbf{i}} u_{\mathbf{i}} = 1$  such that  $\Delta_s \mathbf{u} = \mathbf{u}$ . It follows from the definition of  $\mathbf{u}$  and (2.17) that  $\mathbf{u} = (u_{\mathbf{i}})_{\mathbf{i} \in \mathbb{V}}$  is the unique vector with  $u_{\mathbf{i}} > 0$  for all  $\mathbf{i}$  and  $\sum_{\mathbf{i}} u_{\mathbf{i}} = 1$  such that

$$\Delta \mathbf{u} = \rho \mathbf{u}. \quad (2.18)$$

Define the matrix  $S$  and the vector  $T$  by

$$S = \frac{1}{N\rho} \left( \delta_{\mathbf{i}, \mathbf{j}} u_{\mathbf{i}}^{-1} u_{\mathbf{j}} \right)_{\mathbf{i}, \mathbf{j} \in \mathbb{V}},$$

$$T = \frac{1}{\rho} \left( \sum_{(\mathbf{i}, \mathbf{j}) \in E_{\mathbf{i}}} u_{\mathbf{i}}^{-1} u_{\mathbf{j}} a_{(\mathbf{i}, \mathbf{j})} \right)_{\mathbf{i} \in \mathbb{V}}, \quad (2.19)$$

and put

$$B = (B_{\mathbf{i}})_{\mathbf{i} \in \mathbb{V}} = (1 - S)^{-1} T.$$

Finally, define the vector  $Y$  and the matrix  $M$  by

$$Y = \frac{1}{\rho^2} \left( \sum_{(\mathbf{i}, \mathbf{j}), (\mathbf{i}, \mathbf{k}) \in E_{\mathbf{i}}} u_{\mathbf{i}}^{-2} u_{\mathbf{j}} u_{\mathbf{k}} \left| \left( \frac{1}{N} B_{\mathbf{j}} + a_{(\mathbf{i}, \mathbf{j})} \right) - \left( \frac{1}{N} B_{\mathbf{k}} + a_{(\mathbf{i}, \mathbf{k})} \right) \right| \right)_{\mathbf{i} \in \mathbb{V}}, \quad (2.20)$$

$$M = \frac{1}{N\rho^2} \left( \delta_{\mathbf{i}, \mathbf{j}} u_{\mathbf{i}}^{-2} u_{\mathbf{j}}^2 \right)_{\mathbf{i}, \mathbf{j} \in \mathbb{V}}.$$

We now prove Part 1 and Part 2.

*Part 1. Expressing the average in (2.9) in terms of the  $A_i$ 's and the  $M_i$ 's:* We first equip the index set  $\mathbf{V}$  with the lexicographic order  $\prec$ , say, and define  $s_{\mathbf{i}, \mathbf{j}}$  for  $\mathbf{i}, \mathbf{j} \in \mathbf{V}$  by

$$s_{\mathbf{i}, \mathbf{j}} = \begin{cases} 1 & \text{for } \mathbf{j} \prec \mathbf{i}; \\ -1 & \text{for } \mathbf{i} \prec \mathbf{j}; \\ 0 & \text{for } \mathbf{i} = \mathbf{j}. \end{cases}$$

Next, note that if  $\mathbf{i}, \mathbf{j} \in \mathbf{V}$  with  $\mathbf{i} \neq \mathbf{j}$ , then

$$|x - y| = s_{\mathbf{i}, \mathbf{j}}(x - y) \quad \text{for all } x \in \mathbb{I}_{\mathbf{i}} \text{ and } y \in \mathbb{I}_{\mathbf{j}}. \quad (2.21)$$

Since  $T_N(c, m) = \cup_{\mathbf{i} \in \mathbf{V}} K_{\mathbf{i}}$  (see (2.15)) and  $\mathcal{H}^s(K_{\mathbf{i}} \cap K_{\mathbf{j}}) = 0$  for all  $\mathbf{i}$  and  $\mathbf{j}$  with  $\mathbf{i} \neq \mathbf{j}$  (see (2.16)), we conclude that

$$\begin{aligned} & \frac{1}{\mathcal{H}^s(T_N(c, m))^2} \int_{T_N(c, m)^2} |x - y| d(\mathcal{H}^s \times \mathcal{H}^s)(x, y) \\ &= \frac{1}{\mathcal{H}^s(\cup_{\mathbf{k} \in \mathbf{V}} K_{\mathbf{k}})^2} \int_{\cup_{\mathbf{i}, \mathbf{j} \in \mathbf{V}} (K_{\mathbf{i}} \times K_{\mathbf{j}})} |x - y| d(\mathcal{H}^s \times \mathcal{H}^s)(x, y) \\ &= \sum_{\substack{\mathbf{i}, \mathbf{j} \in \mathbf{V} \\ \mathbf{i} \neq \mathbf{j}}} \frac{1}{\mathcal{H}^s(\cup_{\mathbf{k} \in \mathbf{V}} K_{\mathbf{k}})^2} \int_{K_{\mathbf{i}} \times K_{\mathbf{j}}} |x - y| d(\mathcal{H}^s \times \mathcal{H}^s)(x, y) \\ &= \sum_{\substack{\mathbf{i}, \mathbf{j} \in \mathbf{V} \\ \mathbf{i} \neq \mathbf{j}}} \frac{1}{\mathcal{H}^s(\cup_{\mathbf{k} \in \mathbf{V}} K_{\mathbf{k}})^2} \int_{K_{\mathbf{i}} \times K_{\mathbf{j}}} |x - y| d(\mathcal{H}^s \times \mathcal{H}^s)(x, y) \\ &\quad + \sum_{\mathbf{i} \in \mathbf{V}} \frac{1}{\mathcal{H}^s(\cup_{\mathbf{k} \in \mathbf{V}} K_{\mathbf{k}})^2} \int_{K_{\mathbf{i}}^2} |x - y| d(\mathcal{H}^s \times \mathcal{H}^s)(x, y) \\ &= \sum_{\substack{\mathbf{i}, \mathbf{j} \in \mathbf{V} \\ \mathbf{i} \neq \mathbf{j}}} \frac{1}{\mathcal{H}^s(\cup_{\mathbf{k} \in \mathbf{V}} K_{\mathbf{k}})^2} \int_{K_{\mathbf{i}} \times K_{\mathbf{j}}} s_{\mathbf{i}, \mathbf{j}}(x - y) d(\mathcal{H}^s \times \mathcal{H}^s)(x, y) + \sum_{\mathbf{i} \in \mathbf{V}} \frac{\mathcal{H}^s(K_{\mathbf{i}})^2}{\mathcal{H}^s(\cup_{\mathbf{k} \in \mathbf{V}} K_{\mathbf{k}})^2} A_{\mathbf{i}} \\ &= \sum_{\substack{\mathbf{i}, \mathbf{j} \in \mathbf{V} \\ \mathbf{i} \neq \mathbf{j}}} s_{\mathbf{i}, \mathbf{j}} \left( \frac{1}{\mathcal{H}^s(\cup_{\mathbf{k} \in \mathbf{V}} K_{\mathbf{k}})^2} \int_{K_{\mathbf{i}}} x d\mathcal{H}^s(x) \mathcal{H}^s(K_{\mathbf{j}}) - \frac{1}{\mathcal{H}^s(\cup_{\mathbf{k} \in \mathbf{V}} K_{\mathbf{k}})^2} \int_{K_{\mathbf{j}}} y d\mathcal{H}^s(y) \mathcal{H}^s(K_{\mathbf{i}}) \right) \\ &\quad + \sum_{\mathbf{i} \in \mathbf{V}} \frac{\mathcal{H}^s(K_{\mathbf{i}})^2}{\mathcal{H}^s(\cup_{\mathbf{k} \in \mathbf{V}} K_{\mathbf{k}})^2} A_{\mathbf{i}} \\ &= \sum_{\substack{\mathbf{i}, \mathbf{j} \in \mathbf{V} \\ \mathbf{i} \neq \mathbf{j}}} s_{\mathbf{i}, \mathbf{j}} \left( \frac{\mathcal{H}^s(K_{\mathbf{i}}) \mathcal{H}^s(K_{\mathbf{j}})}{\mathcal{H}^s(\cup_{\mathbf{k} \in \mathbf{V}} K_{\mathbf{k}})^2} M_{\mathbf{i}} - \frac{\mathcal{H}^s(K_{\mathbf{i}}) \mathcal{H}^s(K_{\mathbf{j}})}{\mathcal{H}^s(\cup_{\mathbf{k} \in \mathbf{V}} K_{\mathbf{k}})^2} M_{\mathbf{j}} \right) + \sum_{\mathbf{i} \in \mathbf{V}} \frac{\mathcal{H}^s(K_{\mathbf{i}})^2}{\mathcal{H}^s(\cup_{\mathbf{k} \in \mathbf{V}} K_{\mathbf{k}})^2} A_{\mathbf{i}}. \end{aligned} \quad (2.22)$$

However, it follows from [Wa] (see also [Sp]) that there is a constant  $\lambda$  such that  $\mathcal{H}^s(K_{\mathbf{i}}) = \lambda u_{\mathbf{i}}$  for all  $\mathbf{i} \in \mathbf{V}$ . This and (2.16) imply that  $\frac{\mathcal{H}^s(K_{\mathbf{i}})}{\mathcal{H}^s(\cup_{\mathbf{k} \in \mathbf{V}} K_{\mathbf{k}})} = \frac{\mathcal{H}^s(K_{\mathbf{i}})}{\sum_{\mathbf{k} \in \mathbf{V}} \mathcal{H}^s(K_{\mathbf{k}})} = \frac{\lambda u_{\mathbf{i}}}{\sum_{\mathbf{k} \in \mathbf{V}} \lambda u_{\mathbf{k}}} = \frac{u_{\mathbf{i}}}{\sum_{\mathbf{k} \in \mathbf{V}} u_{\mathbf{k}}} = u_{\mathbf{i}}$  for all  $\mathbf{i}$ , and it therefore follows from (2.22) that

$$\frac{1}{\mathcal{H}^s(T_N(c, m))^2} \int_{T_N(c, m)^2} |x - y| d(\mathcal{H}^s \times \mathcal{H}^s)(x, y) = \sum_{\substack{\mathbf{i}, \mathbf{j} \in \mathbf{V} \\ \mathbf{i} \neq \mathbf{j}}} s_{\mathbf{i}, \mathbf{j}} u_{\mathbf{i}} u_{\mathbf{j}} (M_{\mathbf{i}} - M_{\mathbf{j}}) + \sum_{\mathbf{i} \in \mathbf{V}} u_{\mathbf{i}}^2 A_{\mathbf{i}}. \quad (2.23)$$

Recall, that we write  $\mathbb{I}_{\mathbf{i}} = [s_{\mathbf{i}}, t_{\mathbf{i}}]$ . Using this notation, it is clear that  $M_{\mathbf{i}} = \frac{1}{\mathcal{H}^s(K_{\mathbf{i}})} \int_{K_{\mathbf{i}}} t d\mathcal{H}^s(t) \leq \frac{1}{\mathcal{H}^s(K_{\mathbf{i}})} \int_{K_{\mathbf{i}}} t_{\mathbf{i}} d\mathcal{H}^s(t) = t_{\mathbf{i}}$  and, similarly,  $M_{\mathbf{i}} \geq s_{\mathbf{i}}$ , whence  $M_{\mathbf{i}} \in [s_{\mathbf{i}}, t_{\mathbf{i}}] = \mathbb{I}_{\mathbf{i}}$ . It follows from this and (2.21) that  $s_{\mathbf{i}, \mathbf{j}}(M_{\mathbf{i}} - M_{\mathbf{j}}) = |M_{\mathbf{i}} - M_{\mathbf{j}}|$ , and (2.23) therefore implies that

$$\frac{1}{\mathcal{H}^s(T_N(c, m))^2} \int_{T_N(c, m)^2} |x - y| d(\mathcal{H}^s \times \mathcal{H}^s)(x, y) = \sum_{\mathbf{i}, \mathbf{j} \in \mathbf{V}} u_{\mathbf{i}} u_{\mathbf{j}} |M_{\mathbf{i}} - M_{\mathbf{j}}| + \sum_{\mathbf{i} \in \mathbf{V}} u_{\mathbf{i}}^2 A_{\mathbf{i}}. \quad (2.24)$$

*Part 2. Deriving explicit expressions for  $A_{\mathbf{i}}$  and  $M_{\mathbf{i}}$ :* Since  $\mathbb{I}_{\mathbf{i}} \cap T_N(c, m) = K_{\mathbf{i}}$ , it now follows immediately from Corollary 2.4 and Corollary 2.7 that

$$\begin{aligned} (A_{\mathbf{i}})_{\mathbf{i} \in \mathbf{V}} &= \left( \frac{1}{\mathcal{H}^s(\mathbb{I}_{\mathbf{i}} \cap T_N(c, m))^2} \int_{(\mathbb{I}_{\mathbf{i}} \cap T_N(c, m))^2} |x - y| d(\mathcal{H}^s \times \mathcal{H}^s)(x, y) \right)_{\mathbf{i} \in \mathbf{V}} = (1 - M)^{-1} Y, \\ (M_{\mathbf{i}})_{\mathbf{i} \in \mathbf{V}} &= \left( \frac{1}{\mathcal{H}^s(\mathbb{I}_{\mathbf{i}} \cap T_N(c, m))} \int_{\mathbb{I}_{\mathbf{i}} \cap T_N(c, m)} t d\mathcal{H}^s(t) \right)_{\mathbf{i} \in \mathbf{V}} = (1 - S)^{-1} T. \end{aligned} \quad (2.25)$$

Summarising the results from Part 1 (i.e. (2.24)) and Part 2 (i.e. (2.25)), we have the following formula for the average in between two points in the Drobot-Turner set  $T_N(c, m)$ :

$$\frac{1}{\mathcal{H}^s(T_N(c, m))^2} \int_{T_N(c, m)^2} |x - y| d(\mathcal{H}^s \times \mathcal{H}^s)(x, y) = \sum_{\mathbf{i}, \mathbf{j} \in \mathbf{V}} u_{\mathbf{i}} u_{\mathbf{j}} |M_{\mathbf{i}} - M_{\mathbf{j}}| + \sum_{\mathbf{i} \in \mathbf{V}} u_{\mathbf{i}}^2 A_{\mathbf{i}}, \quad (2.26)$$

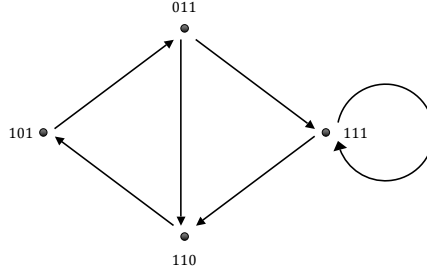
where

$$\begin{aligned} (A_{\mathbf{i}})_{\mathbf{i} \in \mathbf{V}} &= (1 - M)^{-1} Y, \\ (M_{\mathbf{i}})_{\mathbf{i} \in \mathbf{V}} &= (1 - S)^{-1} T. \end{aligned} \quad (2.27)$$

We will now consider a concrete example. Namely we compute the average distance (2.9) between points in the Drobot-Turner set  $T_N(c, m)$  for  $N = 2$ ,  $m = 3$  and  $c = 2$ . If  $N = 2$ ,  $m = 3$  and  $c = 2$ , then we have

$$\begin{aligned} \mathbf{V} &= \{011, 101, 110, 111\}, \\ \mathbf{E} &= \{(011, 110), (011, 111), (111, 111), (111, 110), (110, 101), (101, 011)\}. \end{aligned} \quad (2.28)$$

Figure 1 shows the graph  $G = (\mathbf{V}, \mathbf{E})$  in (2.28).



**Figure 1.** The graph  $G = (\mathbf{V}, \mathbf{E})$  in (2.28) associated with the Drobot-Turner set  $T_2(2, 3)$ .

In the computations below we will always list the entries in the matrices  $\Delta$ ,  $S$  and  $M$  and the vectors  $\mathbf{u}$ ,  $T$  and  $Y$  using the lexicographic order  $\prec$  on the index set  $\mathbf{V}$ ; note that using the lexicographic order  $\prec$  on  $\mathbf{V}$ , we have  $011 \prec 101 \prec 110 \prec 111$ . Using this convention, it is not difficult to see that the matrix  $\Delta$ , the spectral radius  $\rho = \text{spec-rad } \Delta$  and the vector  $\mathbf{u}$  are given by

$$\Delta = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad \rho = \lambda, \quad \mathbf{u} = \frac{1}{2\lambda^2 + \lambda + 1} \begin{pmatrix} \lambda^2 \\ \lambda \\ 1 \\ \lambda^2 \end{pmatrix}, \quad (2.29)$$

where  $\lambda = \frac{1}{3} + \frac{1}{81} \sqrt[3]{\frac{570807 - 59049\sqrt{93}}{2}} + \frac{1}{3} \sqrt[3]{\frac{29 + 3\sqrt{93}}{2}} = 1.465571232 \dots$  is the unique positive real number such that

$$\lambda^3 - \lambda^2 - 1 = 0.$$

We deduce from (2.29) that the vector  $T$  and the matrix  $S$  in (2.19) are given by

$$T = \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ \lambda \\ 1 \end{pmatrix}, \quad S = \frac{1}{2} \begin{pmatrix} 0 & 0 & \frac{1}{\lambda^3} & \frac{1}{\lambda} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\lambda^3} & \frac{1}{\lambda} \end{pmatrix},$$

and so

$$(M_i)_{i \in V} = (B_i)_{i \in V} = (1 - S)^{-1}T = \frac{1}{8\lambda^2 + 14} \begin{pmatrix} 4\lambda^2 + 4\lambda + 2 \\ 2\lambda^2 + 2\lambda + 1 \\ 7\lambda^2 + 8\lambda + 8 \\ 8\lambda^2 + 4\lambda + 9 \end{pmatrix}. \quad (2.30)$$

We also deduce from (2.29) that the vector  $Y$  and the matrix  $M$  in (2.20) are given by

$$Y = \frac{-\lambda^2 + 4\lambda - 1}{38\lambda^2 + 22\lambda + 30} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad M = \frac{1}{2} \begin{pmatrix} 0 & 0 & \frac{1}{\lambda^6} & \frac{1}{\lambda^2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\lambda^6} & \frac{1}{\lambda^2} \end{pmatrix},$$

and so

$$(A_i)_{i \in V} = (1 - M)^{-1}Y = \frac{2\lambda^2 - \lambda + 3}{280\lambda^2 + 108\lambda + 173} \begin{pmatrix} 4 \\ 2 \\ 1 \\ 4 \end{pmatrix}. \quad (2.31)$$

Finally, it follows from (2.26)–(2.27) and (2.29)–(2.31) that

$$\begin{aligned} \frac{1}{\mathcal{H}^s(T_2(2, 3))^2} \int_{T_2(2, 3)^2} |x - y| d(\mathcal{H}^s \times \mathcal{H}^s)(x, y) &= \sum_{i, j \in V} u_i u_j |M_i - M_j| + \sum_{i \in V} u_i^2 A_i \\ &= \frac{4444\lambda^2 + 2071\lambda + 3030}{12141\lambda^2 + 5650\lambda + 8281} \\ &= 0.36610656 \dots \end{aligned}$$

This concludes the example.

### 3. PROOF OF THEOREM 2.1.

The purpose of this section is to prove Theorem 2.1. The main tool for proving Theorem 2.1 is the following well-known result about the asymptotic behaviour of linear difference equations.

**Proposition 3.1.** *Let  $X$  be a Banach space. Let  $T : X \rightarrow X$  be a bounded linear operator with  $\|T\| < 1$  and let  $(y_n)_n$  be a sequence in  $X$  such that  $y_n \rightarrow y$  for some  $y \in X$ . Let the sequence  $(x_n)_n$  be defined by*

$$x_{n+1} = Tx_n + y_n$$

*for all  $n$ . Then  $1 - T$  is invertible and*

$$x_n \rightarrow (1 - T)^{-1}y.$$

Below we will apply Proposition 3.1 to the Banach space given by  $(X, \|\cdot\|) = (\mathbb{R}^V, \|\cdot\|_\infty)$  equipped with the supremum-norm  $\|\cdot\|_\infty$  where  $\|\mathbf{x}\|_\infty = \max_{i \in V} |x_i|$  for  $\mathbf{x} = (x_i)_{i \in V} \in \mathbb{R}^V$ . Bounded linear operators on  $X = \mathbb{R}^V$  are in a natural way identified with  $V \times V$  matrices with real entries, and using this identification, the space of  $V \times V$  matrices with real entries will therefore be equipped with the supremum operator-norm obtained by identifying  $V \times V$  matrices with the linear maps that they induce, i.e. the norm  $\|A\|$  of a  $V \times V$  matrix  $A = (a_{i,j})_{i,j \in V}$  is given by

$$\|A\| = \sup_{\substack{\mathbf{x} \in \mathbb{R}^V \\ \|\mathbf{x}\|_\infty \leq 1}} \|A\mathbf{x}\|_\infty.$$



Below we will also use the following notation. Namely, if  $\mathbf{i} = \mathbf{e}_1 \dots \mathbf{e}_n \in \Sigma_{\mathbf{G}}^n$ , then we will write

$$\begin{aligned} p_{\mathbf{i}} &= p_{\mathbf{e}_1} \cdots p_{\mathbf{e}_n}, \\ q_{\mathbf{i}} &= q_{\mathbf{e}_1} \cdots q_{\mathbf{e}_n}. \end{aligned}$$

Also, recall that we write

$$r_{\mathbf{i}} = r_{\mathbf{e}_1} \cdots r_{\mathbf{e}_n}$$

for all  $\mathbf{i} = \mathbf{e}_1 \dots \mathbf{e}_n \in \Sigma_{\mathbf{G}}^n$ . Using this notation, it is easily seen that  $\mu_{\mathbf{p},\mathbf{i}}(I_{\mathbf{i}}) = p_{\mathbf{i}}$  and  $\mu_{\mathbf{q},\mathbf{i}}(I_{\mathbf{i}}) = q_{\mathbf{i}}$  for  $\mathbf{i} \in \Sigma_{\mathbf{G}}^n$  with  $\mathbf{i}(\mathbf{i}) = \mathbf{i}$ , and it therefore follows that

$$\begin{aligned} A_{n,\mathbf{i}}^{\text{geo}}(\mu_{\mathbf{p},\mathbf{i}}, \mu_{\mathbf{q},\mathbf{i}}) &= \sum_{\substack{\mathbf{i}, \mathbf{j} \in \Sigma_{\mathbf{G}}^n \\ \mathbf{i}(\mathbf{i})=\mathbf{i}(\mathbf{j})=\mathbf{i}}} \frac{\mu_{\mathbf{p},\mathbf{i}}(I_{\mathbf{i}})\mu_{\mathbf{q},\mathbf{i}}(I_{\mathbf{i}})}{r_{\mathbf{i}}r_{\mathbf{j}}} \int_{I_{\mathbf{i}} \times I_{\mathbf{j}}} |x - y| d(x, y) \\ &= \sum_{\substack{\mathbf{i}, \mathbf{j} \in \Sigma_{\mathbf{G}}^n \\ \mathbf{i}(\mathbf{i})=\mathbf{i}(\mathbf{j})=\mathbf{i}}} \frac{p_{\mathbf{i}}q_{\mathbf{j}}}{r_{\mathbf{i}}r_{\mathbf{j}}} \int_{I_{\mathbf{i}} \times I_{\mathbf{j}}} |x - y| d(x, y). \end{aligned}$$

For brevity write

$$A_{n,\mathbf{i}} = A_{n,\mathbf{i}}^{\text{geo}}(\mu_{\mathbf{p}}, \mu_{\mathbf{q}})$$

and define the vector  $A_n$  by

$$A_n = \left( A_{n,\mathbf{i}} \right)_{\mathbf{i} \in \mathbf{V}}.$$

Let  $Y$  and  $M$  be the vector and matrix from Theorem 2.1, i.e.

$$Y = \left( \sum_{\mathbf{e}, \mathbf{f} \in E_{\mathbf{i}}} p_{\mathbf{e}} q_{\mathbf{f}} \left| (r_{\mathbf{e}} B_{\mathbf{p}, \mathbf{t}(\mathbf{e})} + a_{\mathbf{e}}) - (r_{\mathbf{f}} B_{\mathbf{q}, \mathbf{t}(\mathbf{f})} + a_{\mathbf{f}}) \right| \right)_{\mathbf{i} \in \mathbf{V}},$$

and

$$M = \left( \sum_{\mathbf{e} \in E_{\mathbf{i}, \mathbf{j}}} p_{\mathbf{e}} q_{\mathbf{e}} r_{\mathbf{e}} \right)_{\mathbf{i}, \mathbf{j} \in \mathbf{V}}.$$

We must now prove that

$$A_n \rightarrow (1 - M)^{-1} Y.$$

The strategy for proving this is the following. We show that there is a vector  $Y_n = (Y_{n,\mathbf{i}})_{\mathbf{i} \in \mathbf{V}}$  such that

$$\begin{aligned} A_{n+1} &= M A_n + Y_n \quad \text{for all } n, \\ Y_n &\rightarrow Y. \end{aligned} \tag{3.1}$$

Since clearly  $\|M\| < 1$ , it now follows immediately from Proposition 3.1 and (3.1) that

$$A_n \rightarrow (1 - M)^{-1} Y.$$

The proof of (3.1) is divided into into the following 3 parts.

*Part 1 (Section 3.1):* In this section we introduce various auxiliary vectors and prove some auxiliary results that will be needed later.

*Part 2 (Section 3.2):* In this section we construct the vector  $Y_n$  (using the auxiliary matrices and vectors introduced in Section 3.1), and prove that  $Y_n \rightarrow Y$ .

*Part 3 (Section 3.3):* In this section we prove that  $A_{n+1} = M A_n + Y_n$  for all  $n$  and that  $\|M\| < 1$ .

**3.1. The vectors  $B_{\pi,n}$  and  $\lim_n B_{\pi,n}$ .** Fix a family  $\pi = (\pi_e)_{e \in E}$  of positive numbers  $\pi_e$  such that  $(\pi_e)_{e \in E_i}$  is a probability vector for each  $i \in V$ , and for a positive integer  $n$  and  $\mathbf{i} = e_1 \dots e_n \in \Sigma_G^n$ , write  $\pi_{\mathbf{i}} = \pi_{e_1} \dots \pi_{e_n}$ . For each positive integer  $n$  and  $i \in V$ , we now write

$$B_{\pi,n,i} = \sum_{\substack{\mathbf{i} \in \Sigma_G^n \\ i(\mathbf{i})=i}} \frac{\pi_{\mathbf{i}}}{r_{\mathbf{i}}} \int_{I_{\mathbf{i}}} t \, dt, \quad (3.2)$$

and define the vector  $B_{\pi,n}$  by

$$B_{\pi,n} = \left( B_{\pi,n,i} \right)_{i \in V}. \quad (3.3)$$

Also, for each positive integer  $n$  and  $i, j \in V$ , we write

$$\begin{aligned} S_{\pi,i,j} &= \sum_{e \in E_{i,j}} \pi_e r_e, \\ T_{\pi,i} &= \sum_{e \in E_i} \pi_e a_e, \end{aligned} \quad (3.4)$$

and define the matrix  $S_{\pi}$  and the vector  $T_{\pi}$  by

$$\begin{aligned} S_{\pi} &= \left( S_{\pi,i,j} \right)_{i,j \in V}, \\ T_{\pi} &= \left( T_{\pi,i} \right)_{i \in V}; \end{aligned} \quad (3.5)$$

observe that the definitions of the matrix  $S_{\pi}$  and the vector  $T_{\pi}$  are consistent with the definitions of the matrices  $S_{\mathbf{p}}$  and  $S_{\mathbf{q}}$  and the vectors  $T_{\mathbf{p}}$  and  $T_{\mathbf{q}}$  in Theorem 2.1, i.e. if  $\pi = \mathbf{p}$  ( $\pi = \mathbf{q}$ ), then the definitions of  $S_{\pi}$  and  $T_{\pi}$  coincide with the definitions of  $S_{\mathbf{p}}$  and  $T_{\mathbf{p}}$  ( $S_{\mathbf{q}}$  and  $T_{\mathbf{q}}$ ) in Theorem 2.1.

**Lemma 3.2.** *We have  $\|S_{\pi}\| < 1$ .*

*Proof.*

Writing  $r_{\max} = \max_{e \in E} r_e$  and noticing that  $r_{\max} < 1$ , it follows immediately from the definitions of  $S_{\pi}$  and  $\|S_{\pi}\|$  that  $\|S_{\pi}\| = \sup_{(x_i)_{i \in V} \in \mathbb{R}^V, \max_i |x_i| \leq 1} \max_{i \in V} \left| \sum_{j \in V} S_{\pi,i,j} x_j \right| \leq \max_{i \in V} \sum_{j \in V} |S_{\pi,i,j}| = \max_{i \in V} \sum_{j \in V} \sum_{e \in E_{i,j}} \pi_e r_e = \max_{i \in V} \sum_{e \in E_i} \pi_e r_e \leq r_{\max} \max_{i \in V} \sum_{e \in E_i} \pi_e = r_{\max} < 1$  since  $\sum_{e \in E_i} \pi_e = 1$  for all  $i \in V$ .  $\square$

It follows from Lemma 3.2 that  $\|S_{\pi}\| < 1$ , and we therefore conclude from Proposition 3.1 that the matrix  $1 - S_{\pi}$  is invertible. It follows from this that we can define the vector  $B_{\pi}$  by

$$B_{\pi} = (1 - S_{\pi})^{-1} T_{\pi}; \quad (3.6)$$

observe that the definition of the vector  $B_{\pi}$  is consistent with the definitions of the vectors  $B_{\mathbf{p}}$  and  $B_{\mathbf{q}}$  in Theorem 2.1, i.e. if  $\pi = \mathbf{p}$  ( $\pi = \mathbf{q}$ ), then the definition of  $B_{\pi}$  coincides with the definition of  $B_{\mathbf{p}}$  ( $B_{\mathbf{q}}$ ) in Theorem 2.1.

**Proposition 3.3.**

(1) *For all positive integers  $n$ , we have*

$$B_{\pi,n+1} = S_{\pi} B_{\pi,n} + T_{\pi}.$$

(2) *We have*

$$B_{\pi,n} \rightarrow B_{\pi}.$$

*Proof.*

(1) For each vertex  $i \in V$  and all positive integers  $n$ , we have

$$B_{\pi, n+1, i} = \sum_{j \in V} \sum_{e \in E_{i,j}} \sum_{\substack{i \in \Sigma_G^n \\ i(i)=j}} \frac{\pi_e}{r_e} \frac{\pi_i}{r_i} \int_{I_{ei}} t \, dt. \quad (3.7)$$

However, it is clear that if  $e \in E_{i,j}$  and  $i \in \Sigma_G^n$  with  $i(i) = j$ , then  $I_{ei} = S_e I_i$ , whence  $\int_{I_{ei}} t \, dt = \int_{S_e I_i} t \, dt = \int_{I_i} S_e(u) S'_e(u) \, du = \int_{I_i} (r_e u + a_e) r_e \, du$ , and it therefore follows from (3.7) that

$$\begin{aligned} B_{\pi, n+1, i} &= \sum_{j \in V} \sum_{e \in E_{i,j}} \sum_{\substack{i \in \Sigma_G^n \\ i(i)=j}} \frac{\pi_e}{r_e} \frac{\pi_i}{r_i} \int_{I_i} (r_e u + a_e) r_e \, du \\ &= \sum_{j \in V} \left( \sum_{e \in E_{i,j}} \pi_e r_e \right) \left( \sum_{\substack{i \in \Sigma_G^n \\ i(i)=j}} \frac{\pi_i}{r_i} \int_{I_i} u \, du \right) + \sum_{j \in V} \left( \sum_{e \in E_{i,j}} \pi_e a_e \right) \left( \sum_{\substack{i \in \Sigma_G^n \\ i(i)=j}} \frac{\pi_i}{r_i} \int_{I_i} du \right) \\ &= \sum_{j \in V} S_{\pi, i, j} \left( \sum_{\substack{i \in \Sigma_G^n \\ i(i)=j}} \frac{\pi_i}{r_i} \int_{I_i} u \, du \right) + \sum_{j \in V} \left( \sum_{e \in E_{i,j}} \pi_e a_e \right) \left( \sum_{\substack{i \in \Sigma_G^n \\ i(i)=j}} \frac{\pi_i}{r_i} \int_{I_i} du \right) \end{aligned} \quad (3.8)$$

Using the fact that  $\sum_{i \in \Sigma_G^n, i(i)=j} \frac{\pi_i}{r_i} \int_{I_i} u \, du = B_{\pi, n, j}$  and  $\sum_{i \in \Sigma_G^n, i(i)=j} \frac{\pi_i}{r_i} \int_{I_i} du = \sum_{i \in \Sigma_G^n, i(i)=j} \frac{\pi_i}{r_i} = 1$ , we now deduce from (3.8) that

$$\begin{aligned} B_{\pi, n+1, i} &= \sum_{j \in V} S_{\pi, i, j} B_{\pi, n, j} + \sum_{j \in V} \left( \sum_{e \in E_{i,j}} \pi_e a_e \right) \\ &= \sum_{j \in V} S_{\pi, i, j} B_{\pi, n, j} + T_{\pi, i}. \end{aligned}$$

(2) This statement follows immediately from Part (1) and Proposition 3.1.  $\square$

**3.2. The vectors  $Y_n$  and  $\lim_n Y_n$ .** Recall, that the vectors  $B_{\mathbf{p}}$  and  $B_{\mathbf{q}}$  are defined by  $B_{\mathbf{p}} = (1 - S_{\mathbf{p}})^{-1} T_{\mathbf{p}}$  and  $B_{\mathbf{q}} = (1 - S_{\mathbf{q}})^{-1} T_{\mathbf{q}}$  (see (3.6)), and that for each  $i \in V$ , we write  $B_{\mathbf{p}, i}$  and  $B_{\mathbf{q}, i}$  for the  $i$ 'th coordinate of  $B_{\mathbf{p}}$  and  $B_{\mathbf{q}}$ , respectively, i.e. we write

$$B_{\mathbf{p}} = (B_{\mathbf{p}, i})_{i \in V}, \quad B_{\mathbf{q}} = (B_{\mathbf{q}, i})_{i \in V}.$$

For each positive integer  $n$  and  $i \in V$ , we write

$$\begin{aligned} Y_{n, i} &= \sum_{e, f \in E_i} p_e q_f \left| (r_e B_{\mathbf{p}, n, t(e)} + a_e) - (r_f B_{\mathbf{q}, n, t(f)} + a_f) \right|, \\ Y_i &= \sum_{e, f \in E_i} p_e q_f \left| (r_e B_{\mathbf{p}, t(e)} + a_e) - (r_f B_{\mathbf{q}, t(f)} + a_f) \right|. \end{aligned} \quad (3.9)$$

Define the vector  $Y_n$  by

$$Y_n = (Y_{n, i})_{i \in V}, \quad (3.10)$$

and recall that the vector  $Y$  is defined by

$$Y = (Y_i)_{i \in V}.$$

**Proposition 3.4.** *We have*

$$Y_n \rightarrow Y.$$

*Proof.*

This follows immediately from the definitions of  $Y_n$  and  $Y$  since Proposition 3.3 shows that  $B_{\mathbf{p},n} \rightarrow B_{\mathbf{p}}$  and  $B_{\mathbf{q},n} \rightarrow B_{\mathbf{q}}$ .  $\square$

**3.3. The vectors  $A_n$  and  $\lim_n A_n$ .** Recall that for each positive integer  $n$  and each  $\mathbf{i} \in \mathbf{V}$ , we denote the geometric average  $A_{n,\mathbf{i}}^{\text{geo}}(\mu_{\mathbf{p}}, \mu_{\mathbf{q}})$  by  $A_{n,\mathbf{i}}$ , i.e. we write

$$A_{n,\mathbf{i}} = A_{n,\mathbf{i}}^{\text{geo}}(\mu_{\mathbf{p}}, \mu_{\mathbf{q}}) = \sum_{\substack{\mathbf{i}, \mathbf{j} \in \Sigma_{\mathbb{G}}^n \\ \mathbf{i}(\mathbf{i}) = \mathbf{i}(\mathbf{j})}} \frac{p_{\mathbf{i}} q_{\mathbf{j}}}{r_{\mathbf{i}} r_{\mathbf{j}}} \int_{I_{\mathbf{i}} \times I_{\mathbf{j}}} |x - y| d(x, y).$$

Also recall that we define the vector  $A_n$  by

$$A_n = (A_{n,\mathbf{i}})_{\mathbf{i} \in \mathbf{V}}$$

Next, for  $\mathbf{i}, \mathbf{j} \in \mathbf{V}$ , write

$$M_{\mathbf{i},\mathbf{j}} = \sum_{\mathbf{e} \in \mathbf{E}_{\mathbf{i},\mathbf{j}}} p_{\mathbf{e}} q_{\mathbf{e}} r_{\mathbf{e}},$$

and recall that matrix  $M$  is defined by

$$M = (M_{\mathbf{i},\mathbf{j}})_{\mathbf{i},\mathbf{j} \in \mathbf{V}}.$$

Below we show that  $A_{n+1} = M A_n + Y_n$  for all positive integers  $n$ . However, we begin by proving a small lemma.

**Lemma 3.5.** *For all integers  $n$  and all  $\mathbf{i} \in \mathbf{V}$ , we have  $B_{\mathbf{p},n,\mathbf{i}}, B_{\mathbf{q},n,\mathbf{i}} \in I_{\mathbf{i}}$ .*

*Proof.*

Write  $I_{\mathbf{i}} = [\alpha_{\mathbf{i}}, \beta_{\mathbf{i}}]$ . Also, for  $\mathbf{i} \in \Sigma_{\mathbb{G}}^n$  with  $\mathbf{i}(\mathbf{i}) = \mathbf{i}$ , we write  $I_{\mathbf{i}} = [\alpha_{\mathbf{i}}, \beta_{\mathbf{i}}]$  and note that  $[\alpha_{\mathbf{i}}, \beta_{\mathbf{i}}] = I_{\mathbf{i}} = S_{\mathbf{i}}(I_{\mathbf{i}(\mathbf{i})}) \subseteq I_{\mathbf{i}(\mathbf{i})} = I_{\mathbf{i}} = [\alpha_{\mathbf{i}}, \beta_{\mathbf{i}}]$ , whence  $\frac{1}{r_{\mathbf{i}}} \int_{I_{\mathbf{i}}} t dt = \frac{1}{\beta_{\mathbf{i}} - \alpha_{\mathbf{i}}} \int_{\alpha_{\mathbf{i}}}^{\beta_{\mathbf{i}}} t dt = \frac{1}{2(\beta_{\mathbf{i}} - \alpha_{\mathbf{i}})} (\beta_{\mathbf{i}}^2 - \alpha_{\mathbf{i}}^2) = \frac{1}{2} (\alpha_{\mathbf{i}} + \beta_{\mathbf{i}}) \in [\alpha_{\mathbf{i}}, \beta_{\mathbf{i}}] \subseteq [\alpha_{\mathbf{i}}, \beta_{\mathbf{i}}]$ , and so  $\alpha_{\mathbf{i}} \leq \frac{1}{r_{\mathbf{i}}} \int_{I_{\mathbf{i}}} t dt \leq \beta_{\mathbf{i}}$ . Since  $\sum_{\mathbf{i} \in \Sigma_{\mathbb{G}}^n, \mathbf{i}(\mathbf{i}) = \mathbf{i}} p_{\mathbf{i}} = 1$ , this implies that  $B_{\mathbf{p},n,\mathbf{i}} = \sum_{\mathbf{i} \in \Sigma_{\mathbb{G}}^n, \mathbf{i}(\mathbf{i}) = \mathbf{i}} \frac{p_{\mathbf{i}}}{r_{\mathbf{i}}} \int_{I_{\mathbf{i}}} t dt \geq \alpha_{\mathbf{i}} \sum_{\mathbf{i} \in \Sigma_{\mathbb{G}}^n, \mathbf{i}(\mathbf{i}) = \mathbf{i}} p_{\mathbf{i}} = \alpha_{\mathbf{i}}$  and that  $B_{\mathbf{p},n,\mathbf{i}} = \sum_{\mathbf{i} \in \Sigma_{\mathbb{G}}^n, \mathbf{i}(\mathbf{i}) = \mathbf{i}} \frac{p_{\mathbf{i}}}{r_{\mathbf{i}}} \int_{I_{\mathbf{i}}} t dt \leq \beta_{\mathbf{i}} \sum_{\mathbf{i} \in \Sigma_{\mathbb{G}}^n, \mathbf{i}(\mathbf{i}) = \mathbf{i}} p_{\mathbf{i}} = \beta_{\mathbf{i}}$ , i.e.  $B_{\mathbf{p},n,\mathbf{i}} \in [\alpha_{\mathbf{i}}, \beta_{\mathbf{i}}] = I_{\mathbf{i}}$ . A similar argument shows that  $B_{\mathbf{q},n,\mathbf{i}} \in I_{\mathbf{i}}$ .  $\square$

We can now prove that  $A_{n+1} = M A_n + Y_n$  for all positive integers  $n$ .

**Proposition 3.6.**

(1) *For all positive integers  $n$ , we have*

$$A_{n+1} = M A_n + Y_n.$$

(2) *We have  $\|M\| < 1$ .*

(3) *We have*

$$A_n \rightarrow (1 - M)^{-1} Y.$$

*Proof.*

(1) For all positive integers  $n$ , we have

$$A_{n+1,i} = \sum_{e,f \in E_i} \sum_{\substack{\mathbf{i}, \mathbf{j} \in \Sigma_G^n \\ i(\mathbf{i})=t(e) \\ i(\mathbf{j})=t(f)}} \frac{p_e q_f}{r_e r_f} \frac{p_i q_j}{r_i r_j} \int_{I_{ei} \times I_{fj}} |x - y| d(x, y). \quad (3.11)$$

However, it is clear that if  $e, f \in E_i$  and  $\mathbf{i}, \mathbf{j} \in \Sigma_G^n$  with  $i(\mathbf{i}) = t(e)$  and  $i(\mathbf{j}) = t(f)$ , then  $I_{ei} = S_e I_i$  and  $I_{fj} = S_f I_j$ , whence  $\int_{I_{ei} \times I_{fj}} |x - y| d(x, y) = \int_{S_f I_j \times S_e I_i} |x - y| d(x, y) = \int_{I_i \times I_j} |S_e(u) - S_f(v)| r_e r_f d(u, v) = \int_{I_i \times I_j} |(r_e u + a_e) - (r_f v + a_f)| r_e r_f d(u, v)$ , and it therefore follows from (3.11) that

$$\begin{aligned} A_{n+1,i} &= \sum_{e,f \in E_i} \sum_{\substack{\mathbf{i}, \mathbf{j} \in \Sigma_G^n \\ i(\mathbf{i})=t(e) \\ i(\mathbf{j})=t(f)}} \frac{p_e q_f}{r_e r_f} \frac{p_i q_j}{r_i r_j} \int_{I_i \times I_j} |(r_e u + a_e) - (r_f v + a_f)| r_e r_f d(u, v) \\ &= \sum_{\substack{e,f \in E_i \\ e=f}} \sum_{\substack{\mathbf{i}, \mathbf{j} \in \Sigma_G^n \\ i(\mathbf{i})=t(e) \\ i(\mathbf{j})=t(f)}} p_e q_f \frac{p_i q_j}{r_i r_j} \int_{I_i \times I_j} |(r_e u + a_e) - (r_f v + a_f)| d(u, v) \\ &\quad + \sum_{\substack{e,f \in E_i \\ e \neq f}} \sum_{\substack{\mathbf{i}, \mathbf{j} \in \Sigma_G^n \\ i(\mathbf{i})=t(e) \\ i(\mathbf{j})=t(f)}} p_e q_f \frac{p_i q_j}{r_i r_j} \int_{I_i \times I_j} |(r_e u + a_e) - (r_f v + a_f)| d(u, v) \\ &= \sum_{e \in E_i} \sum_{\substack{\mathbf{i}, \mathbf{j} \in \Sigma_G^n \\ i(\mathbf{i})=i(\mathbf{j})=t(e)}} p_e q_e r_e \frac{p_i q_j}{r_i r_j} \int_{I_i \times I_j} |u - v| d(u, v) \\ &\quad + \sum_{\substack{e,f \in E_i \\ e \neq f}} \sum_{\substack{\mathbf{i}, \mathbf{j} \in \Sigma_G^n \\ i(\mathbf{i})=t(e) \\ i(\mathbf{j})=t(f)}} p_e q_f \frac{p_i q_j}{r_i r_j} \int_{I_i \times I_j} |(r_e u + a_e) - (r_f v + a_f)| d(u, v) \\ &= U_{n,i} + V_{n,i}, \end{aligned} \quad (3.12)$$

where

$$\begin{aligned} U_{n,i} &= \sum_{e \in E_i} \sum_{\substack{\mathbf{i}, \mathbf{j} \in \Sigma_G^n \\ i(\mathbf{i})=i(\mathbf{j})=t(e)}} p_e q_e r_e \frac{p_i q_j}{r_i r_j} \int_{I_i \times I_j} |u - v| d(u, v), \\ V_{n,i} &= \sum_{\substack{e,f \in E_i \\ e \neq f}} \sum_{\substack{\mathbf{i}, \mathbf{j} \in \Sigma_G^n \\ i(\mathbf{i})=t(e) \\ i(\mathbf{j})=t(f)}} p_e q_f \frac{p_i q_j}{r_i r_j} \int_{I_i \times I_j} |(r_e u + a_e) - (r_f v + a_f)| d(u, v). \end{aligned}$$

Below we compute  $U_{n,i}$  and  $V_{n,i}$ .

*Claim 1.* We have  $U_{n,i} = \sum_{j \in V} M_{i,j} A_{n,j}$ .

*Proof of Claim 1.* We have

$$\begin{aligned}
U_{n,i} &= \sum_{e \in E_i} \sum_{\substack{\mathbf{i}, \mathbf{j} \in \Sigma_G^n \\ i(\mathbf{i})=i(\mathbf{j})=t(e)}} p_e q_e r_e \frac{p_i q_j}{r_i r_j} \int_{I_i \times I_j} |u - v| d(u, v) \\
&= \sum_{j \in V} \sum_{e \in E_{i,j}} \sum_{\substack{\mathbf{i}, \mathbf{j} \in \Sigma_G^n \\ i(\mathbf{i})=i(\mathbf{j})=j}} p_e q_e r_e \frac{p_i q_j}{r_i r_j} \int_{I_i \times I_j} |u - v| d(u, v) \\
&= \sum_{j \in V} \left( \sum_{e \in E_{i,j}} p_e q_e r_e \right) \left( \sum_{\substack{\mathbf{i}, \mathbf{j} \in \Sigma_G^n \\ i(\mathbf{i})=i(\mathbf{j})=j}} \frac{p_i q_j}{r_i r_j} \int_{I_i \times I_j} |u - v| d(u, v) \right) \\
&= \sum_{j \in V} M_{i,j} A_{n,j}.
\end{aligned}$$

This completes the proof of Claim 1.

*Claim 2.* We have  $V_{n,i} = Y_{n,i}$ .

*Proof of Claim 2.* For  $e, f \in E$ , write

$$s_{e,f} = \begin{cases} 1 & \text{if } r_f + a_f \leq a_e; \\ -1 & \text{if } r_e + a_e \leq a_f; \\ 0 & \text{otherwise.} \end{cases}$$

Since  $S_e(I_{t(e)})^\circ \cap S_f(I_{t(f)})^\circ = \emptyset$  for all  $e, f \in E_i$  with  $e \neq f$ , we conclude that if  $e, f \in E_i$ , then  $|S_e u - S_f v| = s_{e,f}(S_e u - S_f v)$  for all  $u \in I_{t(e)}$  and all  $v \in I_{t(f)}$ , i.e. if  $e, f \in E_i$ , then

$$|(r_e u + a_e) - (r_f v + a_f)| = s_{e,f}((r_e u + a_e) - (r_f v + a_f)) \quad (3.13)$$

for all  $u \in I_{t(e)}$  and all  $v \in I_{t(f)}$ . This and the definition of  $V_{n,i}$  imply that

$$\begin{aligned}
V_{n,i} &= \sum_{\substack{e, f \in E_i \\ e \neq f}} \sum_{\substack{\mathbf{i}, \mathbf{j} \in \Sigma_G^n \\ i(\mathbf{i})=t(e) \\ i(\mathbf{j})=t(f)}} p_e q_f \frac{p_i q_j}{r_i r_j} \int_{I_i \times I_j} s_{e,f}((r_e u + a_e) - (r_f v + a_f)) d(u, v) \\
&= \sum_{e, f \in E_i} \sum_{\substack{\mathbf{i}, \mathbf{j} \in \Sigma_G^n \\ i(\mathbf{i})=t(e) \\ i(\mathbf{j})=t(f)}} p_e q_f \frac{p_i q_j}{r_i r_j} \int_{I_i \times I_j} s_{e,f}((r_e u + a_e) - (r_f v + a_f)) d(u, v) \\
&= \sum_{e, f \in E_i} \sum_{\substack{\mathbf{i}, \mathbf{j} \in \Sigma_G^n \\ i(\mathbf{i})=t(e) \\ i(\mathbf{j})=t(f)}} p_e q_f \frac{p_i q_j}{r_i r_j} s_{e,f}(a_e - a_f) \int_{I_i \times I_j} d(u, v) \\
&\quad + \sum_{e, f \in E_i} \sum_{\substack{\mathbf{i}, \mathbf{j} \in \Sigma_G^n \\ i(\mathbf{i})=t(e) \\ i(\mathbf{j})=t(f)}} p_e q_f \frac{p_i q_j}{r_i r_j} s_{e,f} \int_{I_i \times I_j} (r_e u - r_f v) d(u, v) \\
&= \sum_{e, f \in E_i} \sum_{\substack{\mathbf{i}, \mathbf{j} \in \Sigma_G^n \\ i(\mathbf{i})=t(e) \\ i(\mathbf{j})=t(f)}} p_e q_f p_i q_j s_{e,f}(a_e - a_f) \\
&\quad + \sum_{e, f \in E_i} \sum_{\substack{\mathbf{i}, \mathbf{j} \in \Sigma_G^n \\ i(\mathbf{i})=t(e) \\ i(\mathbf{j})=t(f)}} p_e q_f \frac{p_i q_j}{r_i r_j} s_{e,f} \int_{I_i \times I_j} (r_e u - r_f v) d(u, v). \quad (3.14)
\end{aligned}$$

Using the fact that  $\sum_{\mathbf{i} \in \Sigma_G^n, i(\mathbf{i})=t(\mathbf{e})} p_{\mathbf{i}} = 1$  and  $\sum_{\mathbf{j} \in \Sigma_G^n, i(\mathbf{j})=t(\mathbf{f})} q_{\mathbf{j}} = 1$ , we deduce that

$$\begin{aligned} \sum_{\mathbf{e}, \mathbf{f} \in E_i} \sum_{\substack{\mathbf{i}, \mathbf{j} \in \Sigma_G^n \\ i(\mathbf{i})=t(\mathbf{e}) \\ i(\mathbf{j})=t(\mathbf{f})}} p_{\mathbf{e}} q_{\mathbf{f}} p_{\mathbf{i}} q_{\mathbf{j}} s_{\mathbf{e}, \mathbf{f}} (a_{\mathbf{e}} - a_{\mathbf{f}}) &= \sum_{\mathbf{e}, \mathbf{f} \in E_i} p_{\mathbf{e}} q_{\mathbf{f}} s_{\mathbf{e}, \mathbf{f}} (a_{\mathbf{e}} - a_{\mathbf{f}}) \left( \sum_{\substack{\mathbf{i} \in \Sigma_G^n \\ i(\mathbf{i})=t(\mathbf{e})}} p_{\mathbf{i}} \right) \left( \sum_{\substack{\mathbf{j} \in \Sigma_G^n \\ i(\mathbf{j})=t(\mathbf{f})}} q_{\mathbf{j}} \right) \\ &= \sum_{\mathbf{e}, \mathbf{f} \in E_i} p_{\mathbf{e}} q_{\mathbf{f}} s_{\mathbf{e}, \mathbf{f}} (a_{\mathbf{e}} - a_{\mathbf{f}}). \end{aligned}$$

We conclude from this and (3.14) that

$$\begin{aligned} V_{n,i} &= \sum_{\mathbf{e}, \mathbf{f} \in E_i} p_{\mathbf{e}} q_{\mathbf{f}} s_{\mathbf{e}, \mathbf{f}} (a_{\mathbf{e}} - a_{\mathbf{f}}) + \sum_{\mathbf{e}, \mathbf{f} \in E_i} \sum_{\substack{\mathbf{i}, \mathbf{j} \in \Sigma_G^n \\ i(\mathbf{i})=t(\mathbf{e}) \\ i(\mathbf{j})=t(\mathbf{f})}} p_{\mathbf{e}} q_{\mathbf{f}} \frac{p_{\mathbf{i}} q_{\mathbf{j}}}{r_{\mathbf{i}} r_{\mathbf{j}}} s_{\mathbf{e}, \mathbf{f}} \int_{I_i \times I_j} (r_{\mathbf{e}} u - r_{\mathbf{f}} v) d(u, v) \\ &= \sum_{\mathbf{e}, \mathbf{f} \in E_i} p_{\mathbf{e}} q_{\mathbf{f}} s_{\mathbf{e}, \mathbf{f}} (a_{\mathbf{e}} - a_{\mathbf{f}}) + W_{n,i}, \end{aligned} \quad (3.15)$$

where

$$W_{n,i} = \sum_{\mathbf{e}, \mathbf{f} \in E_i} \sum_{\substack{\mathbf{i}, \mathbf{j} \in \Sigma_G^n \\ i(\mathbf{i})=t(\mathbf{e}) \\ i(\mathbf{j})=t(\mathbf{f})}} p_{\mathbf{e}} q_{\mathbf{f}} \frac{p_{\mathbf{i}} q_{\mathbf{j}}}{r_{\mathbf{i}} r_{\mathbf{j}}} s_{\mathbf{e}, \mathbf{f}} \int_{I_i \times I_j} (r_{\mathbf{e}} u - r_{\mathbf{f}} v) d(u, v).$$

We will now compute  $W_{n,i}$ . In particular, we will express  $W_{n,i}$  in terms of  $B_{\mathbf{p},n}$  and  $B_{\mathbf{q},n}$ . To do so we note that

$$\begin{aligned} W_{n,i} &= \sum_{\mathbf{e}, \mathbf{f} \in E_i} p_{\mathbf{e}} q_{\mathbf{f}} s_{\mathbf{e}, \mathbf{f}} r_{\mathbf{e}} \sum_{\substack{\mathbf{i}, \mathbf{j} \in \Sigma_G^n \\ i(\mathbf{i})=t(\mathbf{e}) \\ i(\mathbf{j})=t(\mathbf{f})}} \frac{p_{\mathbf{i}} q_{\mathbf{j}}}{r_{\mathbf{i}} r_{\mathbf{j}}} \int_{I_i \times I_j} u d(u, v) \\ &\quad - \sum_{\mathbf{e}, \mathbf{f} \in E_i} p_{\mathbf{e}} q_{\mathbf{f}} s_{\mathbf{e}, \mathbf{f}} r_{\mathbf{f}} \sum_{\substack{\mathbf{i}, \mathbf{j} \in \Sigma_G^n \\ i(\mathbf{i})=t(\mathbf{e}) \\ i(\mathbf{j})=t(\mathbf{f})}} \frac{p_{\mathbf{i}} q_{\mathbf{j}}}{r_{\mathbf{i}} r_{\mathbf{j}}} \int_{I_i \times I_j} v d(u, v) \\ &= \sum_{\mathbf{e}, \mathbf{f} \in E_i} p_{\mathbf{e}} q_{\mathbf{f}} s_{\mathbf{e}, \mathbf{f}} r_{\mathbf{e}} \left( \sum_{\substack{\mathbf{i} \in \Sigma_G^n \\ i(\mathbf{i})=t(\mathbf{e})}} \frac{p_{\mathbf{i}}}{r_{\mathbf{i}}} \int_{I_i} u du \right) \left( \sum_{\substack{\mathbf{j} \in \Sigma_G^n \\ i(\mathbf{j})=t(\mathbf{f})}} \frac{q_{\mathbf{j}}}{r_{\mathbf{j}}} \int_{I_j} dv \right) \\ &\quad - \sum_{\mathbf{e}, \mathbf{f} \in E_i} p_{\mathbf{e}} q_{\mathbf{f}} s_{\mathbf{e}, \mathbf{f}} r_{\mathbf{f}} \left( \sum_{\substack{\mathbf{i} \in \Sigma_G^n \\ i(\mathbf{i})=t(\mathbf{e})}} \frac{p_{\mathbf{i}}}{r_{\mathbf{i}}} \int_{I_i} du \right) \left( \sum_{\substack{\mathbf{j} \in \Sigma_G^n \\ i(\mathbf{j})=t(\mathbf{f})}} \frac{q_{\mathbf{j}}}{r_{\mathbf{j}}} \int_{I_j} v dv \right). \end{aligned} \quad (3.16)$$

Using the fact that  $\sum_{\mathbf{i} \in \Sigma_G^n, i(\mathbf{i})=t(\mathbf{e})} \frac{p_{\mathbf{i}}}{r_{\mathbf{i}}} \int_{I_i} u du = B_{\mathbf{p},n,t(\mathbf{e})}$  and  $\sum_{\mathbf{j} \in \Sigma_G^n, i(\mathbf{j})=t(\mathbf{f})} \frac{q_{\mathbf{j}}}{r_{\mathbf{j}}} \int_{I_j} v dv = B_{\mathbf{q},n,t(\mathbf{f})}$ , together with the fact that  $\sum_{\mathbf{i} \in \Sigma_G^n, i(\mathbf{i})=t(\mathbf{e})} \frac{p_{\mathbf{i}}}{r_{\mathbf{i}}} \int_{I_i} du = \sum_{\mathbf{i} \in \Sigma_G^n, i(\mathbf{i})=t(\mathbf{e})} \frac{p_{\mathbf{i}}}{r_{\mathbf{i}}} r_{\mathbf{i}} = \sum_{\mathbf{i} \in \Sigma_G^n, i(\mathbf{i})=t(\mathbf{e})} p_{\mathbf{i}} = 1$  and  $\sum_{\mathbf{j} \in \Sigma_G^n, i(\mathbf{j})=t(\mathbf{f})} \frac{q_{\mathbf{j}}}{r_{\mathbf{j}}} \int_{I_j} dv = \sum_{\mathbf{j} \in \Sigma_G^n, i(\mathbf{j})=t(\mathbf{f})} \frac{q_{\mathbf{j}}}{r_{\mathbf{j}}} r_{\mathbf{j}} = \sum_{\mathbf{j} \in \Sigma_G^n, i(\mathbf{j})=t(\mathbf{f})} q_{\mathbf{j}} = 1$ , it follows from (3.16) that

$$W_{n,i} = \sum_{\mathbf{e}, \mathbf{f} \in E_i} p_{\mathbf{e}} q_{\mathbf{f}} s_{\mathbf{e}, \mathbf{f}} r_{\mathbf{e}} B_{\mathbf{p},n,t(\mathbf{e})} - \sum_{\mathbf{e}, \mathbf{f} \in E_i} p_{\mathbf{e}} q_{\mathbf{f}} s_{\mathbf{e}, \mathbf{f}} r_{\mathbf{f}} B_{\mathbf{q},n,t(\mathbf{f})} \quad (3.17)$$

Finally, combining (3.15) and (3.17) shows that

$$\begin{aligned} V_{n,i} &= \sum_{\mathbf{e}, \mathbf{f} \in E_i} p_{\mathbf{e}} q_{\mathbf{f}} s_{\mathbf{e}, \mathbf{f}} (a_{\mathbf{e}} - a_{\mathbf{f}}) + W_{n,i} \\ &= \sum_{\mathbf{e}, \mathbf{f} \in E_i} p_{\mathbf{e}} q_{\mathbf{f}} s_{\mathbf{e}, \mathbf{f}} (a_{\mathbf{e}} - a_{\mathbf{f}}) + \sum_{\mathbf{e}, \mathbf{f} \in E_i} p_{\mathbf{e}} q_{\mathbf{f}} s_{\mathbf{e}, \mathbf{f}} r_{\mathbf{e}} B_{\mathbf{p},n,t(\mathbf{e})} - \sum_{\mathbf{e}, \mathbf{f} \in E_i} p_{\mathbf{e}} q_{\mathbf{f}} s_{\mathbf{e}, \mathbf{f}} r_{\mathbf{f}} B_{\mathbf{q},n,t(\mathbf{f})} \\ &= \sum_{\mathbf{e}, \mathbf{f} \in E_i} p_{\mathbf{e}} q_{\mathbf{f}} s_{\mathbf{e}, \mathbf{f}} ((r_{\mathbf{e}} B_{\mathbf{p},n,t(\mathbf{e})} + a_{\mathbf{e}}) - (r_{\mathbf{f}} B_{\mathbf{q},n,t(\mathbf{f})} + a_{\mathbf{f}})). \end{aligned} \quad (3.18)$$

However, since it follows from Lemma 3.5 that  $B_{\mathbf{p},n,t(\mathbf{e})} \in I_{t(\mathbf{e})}$  and  $B_{\mathbf{q},n,t(\mathbf{f})} \in I_{t(\mathbf{f})}$ , we conclude from (3.13) that  $s_{\mathbf{e},\mathbf{f}}((r_{\mathbf{e}}B_{\mathbf{p},n,t(\mathbf{e})} + a_{\mathbf{e}}) - (r_{\mathbf{f}}B_{\mathbf{q},n,t(\mathbf{f})} + a_{\mathbf{f}})) = |(r_{\mathbf{e}}B_{\mathbf{p},n,t(\mathbf{e})} + a_{\mathbf{e}}) - (r_{\mathbf{f}}B_{\mathbf{q},n,t(\mathbf{f})} + a_{\mathbf{f}})|$ , and (3.17) and the definition of  $Y_{n,i}$  (see (3.9)) therefore show that

$$\begin{aligned} V_{n,i} &= \sum_{\mathbf{e},\mathbf{f} \in E_i} p_{\mathbf{e}} q_{\mathbf{f}} |(r_{\mathbf{e}}B_{\mathbf{p},n,t(\mathbf{e})} + a_{\mathbf{e}}) - (r_{\mathbf{f}}B_{\mathbf{q},n,t(\mathbf{f})} + a_{\mathbf{f}})| \\ &= Y_{n,i}. \end{aligned}$$

This completes the proof of Claim 2.

It follows immediately from (3.12), Claim 1 and Claim 2 that  $A_{n+1,i} = U_{n,i} + V_{n,i} = \sum_{j \in V} M_{i,j} A_{n,j} + Y_{n,i}$ , i.e.  $A_{n+1} = MA_n + Y$ .

(2) Using the same notation as in the proof of Lemma 3.2, we write  $r_{\max} = \max_{\mathbf{e} \in E} r_{\mathbf{e}}$  and note that  $r_{\max} < 1$ . It now follows immediately from the definitions of the matrix  $M$  and  $\|M\|$  that  $\|M\| = \sup_{(x_i)_{i \in V} \in \mathbb{R}^V, \max_i |x_i| \leq 1} \max_{i \in V} |\sum_{j \in V} M_{i,j} x_j| \leq \max_{i \in V} \sum_{j \in V} |M_{i,j}| = \max_{i \in V} \sum_{j \in V} \sum_{\mathbf{e} \in E_{i,j}} p_{\mathbf{e}} q_{\mathbf{e}} r_{\mathbf{e}} = \max_{i \in V} \sum_{\mathbf{e} \in E_i} p_{\mathbf{e}} q_{\mathbf{e}} r_{\mathbf{e}} \leq r_{\max} \max_{i \in V} \sum_{\mathbf{e}, \mathbf{f} \in E_i} p_{\mathbf{e}} q_{\mathbf{f}} = r_{\max} (\sum_{\mathbf{e} \in E_i} p_{\mathbf{e}}) (\sum_{\mathbf{f} \in E_i} q_{\mathbf{f}}) = r_{\max} < 1$  since, by assumption,  $\sum_{\mathbf{e} \in E_i} p_{\mathbf{e}} = \sum_{\mathbf{f} \in E_i} q_{\mathbf{f}} = 1$  for all  $i \in V$ .

(3) This statement follows immediately from Part (1), Part (2) and Proposition 3.1 since Proposition 3.4 shows that  $Y_n \rightarrow Y$ .  $\square$

We can now prove Theorem 2.1.

*Proof of Theorem 2.1.*

Let  $M$  and  $Y$  be as in Theorem 2.1. Since

$$(A_{n,i}^{\text{geo}}(\mu_{\mathbf{p},i}, \mu_{\mathbf{q},i}))_{i \in V} = (A_{n,i})_{i \in V} = A_n,$$

it follows from Proposition 1.1 and Proposition 3.6.(2) that

$$(A_i(\mu_{\mathbf{p},i}, \mu_{\mathbf{q},i}))_{i \in V} = (A_i^{\text{geo}}(\mu_{\mathbf{p},i}, \mu_{\mathbf{q},i}))_{i \in V} = \lim_n (A_{n,i}^{\text{geo}}(\mu_{\mathbf{p},i}, \mu_{\mathbf{q},i}))_{i \in V} = \lim_n A_n = (1 - M)^{-1}Y.$$

This completes the proof of Theorem 2.1.  $\square$

#### 4. PROOF OF THEOREM 2.6.

The purpose of this section is to prove Theorem 2.6.

*Proof of Theorem 2.6.*

Let  $S$  and  $T$  be as in Theorem 2.6, and let  $S_{\mathbf{p}}$ ,  $T_{\mathbf{p}}$  and  $B_{\mathbf{p}} = (1 - S_{\mathbf{p}})^{-1}T_{\mathbf{p}}$  be as in (3.5) and (3.6), respectively, and note that  $S = S_{\mathbf{p}}$  and  $T = T_{\mathbf{p}}$ . In particular, this implies that

$$(1 - S)^{-1} = (1 - S_{\mathbf{p}})^{-1}T_{\mathbf{p}} = B_{\mathbf{p}}. \quad (3.19)$$

Next, for each positive integer  $n$  and  $i \in V$ , let  $B_{\mathbf{p},n,i}$  be defined as in (3.2), i.e.

$$B_{\mathbf{p},n,i} = \sum_{\substack{\mathbf{i} \in \Sigma_{\mathbf{G}}^n \\ i(\mathbf{i})=i}} \frac{p_{\mathbf{i}}}{r_{\mathbf{i}}} \int_{I_{\mathbf{i}}} t \, dt,$$

and let the vector  $B_{\mathbf{p},n}$  be define as in (3.2), i.e.

$$B_{\mathbf{p},n} = (B_{\mathbf{p},n,i})_{i \in V}.$$

For  $\mathbf{i} \in \Sigma_{\mathbf{G}}^n$ , let  $\lambda_{\mathbf{i}}$  denote the normalized Lebesgue measure restricted to  $I_{\mathbf{i}}$ . Next, for a positive integer  $n$ , define the measure  $\tilde{\mu}_{\mathbf{p},n,i}$  by  $\tilde{\mu}_{\mathbf{p},n,i} = \sum_{\mathbf{i} \in \Sigma_{\mathbf{G}}^n, i(\mathbf{i})=i} \mu(I_{\mathbf{i}}) \lambda_{\mathbf{i}} = \sum_{\mathbf{i} \in \Sigma_{\mathbf{G}}^n, i(\mathbf{i})=i} p_{\mathbf{i}} \lambda_{\mathbf{i}}$ . It is not



difficult to see that  $\tilde{\mu}_{\mathbf{p},n,i} \rightarrow \mu_{\mathbf{p},i}$  weakly. In particular, since clearly  $B_{\mathbf{p},n,i} = \int t d\tilde{\mu}_{\mathbf{p},n,i}(t)$  and  $M_i(\mu_{\mathbf{p},i}) = \int t d\mu_{\mathbf{p},i}(t)$ , this implies that  $B_{\mathbf{p},n,i} = \int t d\tilde{\mu}_{\mathbf{p},n,i}(t) \rightarrow \int t d\mu_{\mathbf{p},i}(t) = M_i(\mu_{\mathbf{p},i})$ . Hence


$$\left(M_i(\mu_{\mathbf{p},i})\right)_{i \in V} = \lim_n B_{\mathbf{p},n},$$

and it therefore follows from Proposition 3.3 and (3.19) that

$$\left(M_i(\mu_{\mathbf{p},i})\right)_{i \in V} = \lim_n B_{\mathbf{p},n} = B_{\mathbf{p}} = (1 - S)^{-1}T.$$

This completes the proof.

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